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## Perturbative Couplings of Vector Multiplets in $N = 2$ Heterotic String Vacua\*

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### ABSTRACT

We study the low-energy effective Lagrangian of  $N = 2$  heterotic string vacua at the classical and quantum level. The couplings of the vector multiplets are uniquely determined at the tree level, while the loop corrections are severely constrained by the exact discrete symmetries of the string vacuum. We evaluate the general transformation law of the perturbative prepotential and determine its form for the toroidal compactifications of six-dimensional  $N = 1$  supersymmetric vacua.

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# 1 Introduction and Summary

Four-dimensional gauge theories invariant under  $N = 2$  supersymmetry have revealed several interesting features about (supersymmetric) quantum field theories, although they themselves are not directly related to physical phenomena at or below the electro-weak scale. In part these features are due to the strong restrictions imposed by  $N = 2$  supersymmetry on the couplings of the classical Lagrangian and its possible counterterms at the quantum level. This fact leads to a number of non-renormalization theorems which are usually much stronger than their  $N = 1$  counterparts [1, 2]. Recently, Seiberg and Witten [3] were able to solve rigidly  $N = 2$  supersymmetric Yang-Mills theory with a gauge group  $SU(2)$  using the analytic properties of the  $N = 2$  couplings. This led to exciting insight into non-perturbative phenomena of quantum field theories, such as confinement and monopole condensation.

It is of interest to consider the analysis of Seiberg and Witten and its generalizations to larger gauge groups [4] in the context of  $N = 2$  string theories. Locally  $N = 2$  supersymmetric gauge theories arise in four space-time dimensions either as type-II string vacua or as vacua of the heterotic string. In all type-II vacua the size of the gauge group is severely limited [5], while in the heterotic string this constraint is much weaker and large gauge groups typically appear at certain points in the classical moduli space. Therefore, the latter provides the suitable framework to study non-trivial gauge dynamics à la Seiberg and Witten. However, in this paper we solely focus on the perturbative properties of  $N = 2$  heterotic string vacua as a first step in this direction.

In section 2 we briefly summarize known properties of the four-dimensional  $N = 2$  heterotic strings. From the world-sheet point of view, the heterotic vacua do not have left-right symmetry: their left-moving degrees of freedom are described in terms of a bosonic conformal field theory (CFT) with central charge  $\bar{c} = 22$ , while the right-moving sector is build out of a free superconformal field theory (SCFT) with  $c = 3$  and an interacting SCFT with  $N = 4$  world-sheet supersymmetry and central charge  $c = 6$  [6]. In space-time, the massless spectrum of such a vacuum always contains the  $N = 2$  gravitational multiplet consisting of the graviton, two gravitinos and the graviphoton — a spin-1 Abelian gauge boson. The other gauge bosons are members of  $N = 2$  vector multiplets, which also contain two gauginos and a complex scalar field, both in the adjoint representation of the gauge group  $G$ . The matter fields which are charged under the gauge group (usually in the fundamental representation of  $G$ ) reside in  $N = 2$  hypermultiplets. Most string vacua also contain gauge-neutral moduli scalars, which correspond to exact flat directions of the effective potential. Their vacuum expectation values are not determined at the perturbative level and are thus free parameters of the string vacua. In  $N = 2$  theories such moduli can arise in either vector or hypermultiplets.

The dilaton, the antisymmetric tensor and a vector boson are contained in a special  $N = 2$  moduli multiplet. In appendix A we display the linearized transformation rules of this new  $N = 2$  multiplet, called vector-tensor multiplet, as well as its free action, which describes the appropriate degrees of freedom. It is constructed out of an  $N = 1$  vector

multiplet and an  $N = 1$  linear multiplet and its superalgebra necessarily has an off-shell central charge. In four space-time dimensions an antisymmetric tensor is always dual to a scalar field (the axion) and thus the dilaton/axion can be put into an Abelian vector multiplet. However, at present we can perform the duality transformation only at the component level, without preserving the off-shell supersymmetry.\*

Section 3 is devoted to locally  $N = 2$  supersymmetric field theories at the classical and quantum level. The couplings of the vector multiplets are encoded in a single holomorphic prepotential  $F$  that is homogeneous of degree two. The  $\sigma$ -model metric for the scalar fields is the metric of a ‘special Kähler manifold’ in that its Kähler potential is determined by the holomorphic  $F$  [9]; the gauge couplings follow from the second derivatives of  $F$ . The global symmetries of the theory, continuous or discrete, have to respect the special properties of the vector couplings and therefore have to constitute a subgroup of  $Sp(2n + 2)$ , where  $n$  is the number of vector multiplets.

In section 3.1 we briefly review the couplings of the vector multiplets and their transformation properties; this will be important for discussing the discrete quantum symmetries of the heterotic string in section 4. We draw attention to the fact that in certain parametrizations a prepotential does not exist, a phenomenon discussed recently in [10]; the relevance of such parametrizations is discussed later. In section 3.2 we study quantum properties of  $N = 2$  theories. Just as in  $N = 1$  supersymmetric quantum field theories, it is essential to distinguish the Wilsonian couplings of the effective theory from the physical, momentum-dependent effective couplings [11, 12]. The former share all the analytic properties of their classical counterparts; in particular, they are determined from the loop corrections to the holomorphic prepotential  $F$ . But in theories with massless charged fields, the momentum-dependent effective gauge couplings are different from the Wilsonian couplings and do not share their analytic properties. The difference between the two kinds of couplings can be computed entirely within the low-energy effective field theory [13, 12, 14]; for  $N = 2$  this computation is outlined in section 3.2.

In any heterotic string vacuum, the dilaton enjoys very special properties. To all orders in perturbation theory it has a continuous Peccei-Quinn symmetry, and at the tree level, it is completely orthogonal to the other moduli scalars. According to a theorem by Ferrara and Van Proeyen [15], these two requirements uniquely determine the *tree-level* prepotential  $F$  for all heterotic  $N = 2$  vacua. The properties of this prepotential, in particular its target-space symmetries, are discussed in section 4.1. (Most of these tree-level results were independently obtained in ref. [10]).

The loop corrections to the prepotential are severely constrained by the Peccei-Quinn symmetry, which prohibits any higher-loop corrections, and by the exact target-space duality symmetries of the string vacua. In section 4.2 we generalize the formalism of section 3.1 to include quantum corrections and determine the general transformation law of the loop-corrected  $F$ .

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\*Note that in type-II theories the dilaton and the antisymmetric tensor reside in an  $N = 2$  tensor multiplet, which is dual to an  $N = 2$  hypermultiplet [7, 8].

Finally, in section 4.3 we apply this formalism to the concrete case of  $N = 2$  vacua that arise as toroidal compactifications of six-dimensional  $N = 1$  theories. This class of vacua always features the two toroidal moduli  $T$  and  $U$  and an exact invariance under the modular group  $[SL(2, \mathbf{Z})]^2$ . For this case the modular forms are known and the transformation laws derived in section 4.2 completely determine the loop-corrected prepotential (up to a quadratic polynomial that may add field-independent constants to the gauge couplings). We also discuss what happens when a toroidal compactification is deformed by a Wilson-line modulus.

## 2 $N = 2$ Vacua of the Heterotic String

In this section we recall some of the known features of four-dimensional  $N = 2$  heterotic string vacua which are needed for our later discussions. Further details can be found in the literature, for example, in refs. [6, 16].

One way to characterize different string vacua is via their underlying two-dimensional (super)conformal field theories (SCFTs) on the world sheet. As we have already mentioned, the SCFTs of the heterotic vacua are not left-right symmetric and their left-moving and right-moving world-sheet degrees of freedom are quite distinct from each other. The right-moving side has a local world-sheet supersymmetry and consists of four free bosons  $X^\mu(z)$ , which, along with four free fermions  $\psi^\mu(z)$ , generate the four-dimensional space-time. Together with the superconformal ghosts  $b(z)$ ,  $c(z)$ ,  $\beta(z)$  and  $\gamma(z)$ , they contribute  $-9$  units to the central charge which has to be balanced by the central charge of an appropriate “internal” SCFT. This internal SCFT is further constrained by the desired space-time properties of the string vacuum, in particular by the amount of space-time supersymmetry.

$N$ -extended space-time supersymmetry implies the existence of  $N$  supercharges  $Q_\alpha^i$  ( $i = 1, \dots, N$ ) obeying the supersymmetry algebra in four-dimensional Minkowski space:

$$\begin{aligned} \{Q_\alpha^i, \bar{Q}_{\dot{\beta}j}\} &= 2i\delta_j^i \gamma_{\alpha\dot{\beta}}^\mu P_\mu, \\ \{Q_\alpha^i, Q_\beta^j\} &= 2C_{\alpha\beta} Z^{ij}, \end{aligned} \tag{2.1}$$

where  $Z^{ij}$  denotes the central charges.  $\alpha$  and  $\dot{\beta}$  are two-component spinor indices associated with chiral spinors in a four-dimensional space-time, and  $C_{\alpha\beta}$  is the charge-conjugation matrix. In the heterotic string theory, all the supercharges ‘live’ on the right-moving (supersymmetric) side of the SCFT and thus are defined in terms of holomorphic local two-dimensional operators given by the holomorphic parts of the gravitino vertex operators  $V_\alpha^i$  and  $\bar{V}_{\dot{\alpha}i}$  at zero momentum:

$$Q_\alpha^i = \oint \frac{dz}{2\pi i} V_\alpha^i(z), \quad \bar{Q}_{\dot{\alpha}i} = \oint \frac{dz}{2\pi i} \bar{V}_{\dot{\alpha}i}(z), \tag{2.2}$$

where  $V_\alpha^i(z)$  and  $\bar{V}_{\dot{\alpha}i}(z)$  (in the  $-1/2$  ghost picture) are given by

$$V_\alpha^i(z) = S_\alpha \Sigma^i e^{-\frac{\phi}{2}}(z), \quad \bar{V}_{\dot{\alpha}i}(z) = S_{\dot{\alpha}} \Sigma_i e^{-\frac{\phi}{2}}(z). \tag{2.3}$$

Here  $S_\alpha(z)$ ,  $S_{\dot{\alpha}}(z)$  are four-dimensional spin fields and  $\phi(z)$  originates from the bosonization of the superconformal ghosts fields  $\beta$  and  $\gamma$ .  $\Sigma^i(z)$  and  $\Sigma_i(z)$  are conformal fields of the  $c = 9$  internal SCFT and have conformal dimension  $3/8$ . For heterotic vacua the  $N = 2$  space-time supersymmetry charges reside entirely in the right-moving sector. This is in contrast with the type-II string, where one has one right-moving and one left-moving supercharge operator.

For a heterotic string vacuum with  $N = 2$  space-time supersymmetry the algebra (2.1) implies that the internal right-moving  $c = 9$  SCFT splits into a  $c = 3$  SCFT with  $N = 2$  world-sheet supersymmetry and a  $c = 6$  piece with  $N = 4$  supersymmetry on the world-sheet [6, 16]. The  $c = 3$  system can be realized by a free complex  $N = 1$  superfield whose bosonic components we denote by  $\partial X^\pm(z)$ , while the complex fermions are bosonized according to  $\psi^\pm(z) = \exp(\pm iH(z))$  with  $H(z)$  a free real scalar field. On the other hand, the  $N = 4, c = 6$  SCFT is not free, although it necessarily contains a free boson  $J^3(z) = \frac{i}{\sqrt{2}}\partial H'(z)$  corresponding to the Cartan generator of the level-1  $SU(2)$  Kač-Moody algebra of the  $N = 4$  theory. In terms of  $H$  and  $H'$  the internal part of the gravitino vertex operators can be represented by  $(\Sigma_j(z) = (\Sigma^j)^\dagger)$ :

$$\Sigma^j(z) = \exp\left(\frac{i}{2}H(z)\right) \exp\left((-)^{j+1}\frac{i}{\sqrt{2}}H'(z)\right). \quad (2.4)$$

Evaluating the operator product of  $\Sigma^i(z)$  with  $\Sigma^j(w)$  one finds the central-charge operator of the  $N = 2$  algebra to be

$$Z^\pm = \oint \frac{dz}{2\pi i} \psi^\pm(z) e^{-\phi(z)}. \quad (2.5)$$

In the zero-ghost picture  $Z^\pm$  is proportional to  $\partial X^\pm(z)$  and hence the central charge of a massive state is determined by the momentum eigenvalues  $(p^+, p^-)$  in the  $c = 3$  SCFT. This implies that vertex operators for massive states with non-vanishing central charges contain factors of the form  $\exp(ip^+X^-(z) + ip^-X^+(z))$ .

Generally, there is no world-sheet supersymmetry on the left-moving side of the heterotic string, which is based on an ordinary bosonic CFT with central charge  $\bar{c} = 26$ . This CFT is comprised of a  $\bar{c} = 4$  sector containing four free bosons  $X^\mu(\bar{z})$ , which generate the four-dimensional space-time, and of an arbitrary  $\bar{c} = 22$  sector.

The massless spectrum of a heterotic  $N = 2$  vacuum always comprises the graviton ( $G_{\mu\nu}$ ), the antisymmetric tensor ( $B_{\mu\nu}$ ) and the dilaton ( $D$ ), created by vertex operators of the form  $\bar{\partial}X_\mu(\bar{z})\partial X_\nu(z)$  (at zero momentum), and two gravitini and two dilatini, created by vertex operators  $\bar{\partial}X_\mu(\bar{z})V_\alpha^i(z)$ . In addition, there are always two Abelian gauge bosons  $A_\mu^\pm$  with vertex operators  $\bar{\partial}X_\mu(\bar{z})\partial X^\pm(z)$ , which generate the gauge group  $[U(1)_R]^2$  (the suffix  $R$  indicates that these groups originate from the dimension-one operators  $\partial X^\pm(z)$  of the right-moving sector). One linear combination is the graviphoton, which is the spin-1 gauge boson of the  $N = 2$  supergravity multiplet that also contains the graviton and two gravitini.\* The dilaton together with the antisymmetric tensor  $B_{\mu\nu}$ , the two dilatini and

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\*Note that the  $\partial X^\pm(z)$  factors of the vertex operators of the Abelian gauge bosons coincide with the central-charge operator and thus the charges of  $[U(1)_R]^2$  are identical to the central charges  $p^+, p^-$ .

the remaining  $U(1)$  vector are naturally described by an  $N = 2$  *vector-tensor* multiplet. As far as we know, this type of an  $N = 2$  supermultiplet has not been discussed in the previous literature. In appendix A we establish the linearized transformation properties of this new vector-tensor multiplet and display its free action. In a dual description, where the antisymmetric tensor is replaced by a pseudo-scalar (axion)  $a$ , the degrees of freedom form a  $N = 2$  vector supermultiplet where the dilaton and the axion combine into a complex scalar  $S = e^D + ia$ .<sup>†</sup> Note that the ‘heterotic’ vector-tensor multiplet is different from the standard  $N = 2$  tensor multiplet [8], which describes the dilaton multiplet in type-II  $N = 2$  theories. The ‘type-II’ tensor multiplet contains no vector boson but has instead two additional Ramond-Ramond scalars and is thus dual to an  $N = 2$  hypermultiplet [7].

Apart from the two Abelian gauge bosons we just discussed, the massless spectrum of heterotic string vacua contain further gauge bosons  $A_\mu^a$ , which are always members of  $N = 2$  vector multiplets. Their superpartners are two gaugini  $\lambda_{i\alpha}^a$  and a complex scalar  $C^a$  and the vertex operators for a generic vector multiplet are given by

$$\left(A_\mu^a, \lambda_{i\alpha}^a, C^a\right) \sim \left(J^a(\bar{z}) \partial X_\mu(z), J^a(\bar{z}) V_{i\alpha}(z), J^a(\bar{z}) \partial X^\pm(z)\right), \quad (2.6)$$

where  $J^a(\bar{z})$  are dimension  $(1,0)$  operators that together comprise a left-moving Kač-Moody current algebra. Their zero modes generate a non-Abelian gauge group  $G$ ; the currents  $J^a$  and hence the entire corresponding vector multiplets transform in the adjoint representation of this group. The maximal rank of  $G$  is bounded by the central charge  $\bar{c}$  of the Kač-Moody algebra and therefore cannot exceed 22. Furthermore,  $G$  is not necessarily a simple group, but rather contains several simple and/or Abelian factors. For example, the compactification of the ten-dimensional heterotic string on a six-dimensional manifold  $T^2 \times K_3$  leads to the gauge group  $E_7 \times E_8 \times U(1)_L^2 \times U(1)_R^2$ <sup>‡</sup> and the world-sheet supersymmetries are maximally extended to  $(2,2) \oplus (4,4)$ . Altogether, this class of vacuum families has  $\dim(E_7) + \dim(E_8) + 2 + 2 = 385$  gauge fields and the corresponding low-energy effective  $N = 2$  supergravity has 384 vector supermultiplets (including the Abelian vector multiplet for the dilaton but excluding the graviphoton).

The scalar fields in the Cartan subalgebras of non-Abelian factors  $G_{(a)} \subset G$  as well as the scalars of any Abelian factor in  $G$  correspond to flat directions of the  $N = 2$  scalar potential. Their vertex operators are truly marginal operators of the SCFT and the corresponding space-time vacuum expectation values are free parameters which continuously connect a family of string vacua. However, the low-energy description of the vacuum family depends on the size of these vacuum expectation values. If they vanish, the entire non-Abelian gauge group is intact and the associated massless gauge bosons appear in the low-energy effective Lagrangian. On the other hand, non-vanishing vacuum expectation values of flat directions would spontaneously break the gauge symmetry

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<sup>†</sup>At present we can perform the duality transformation only at the component level.

<sup>‡</sup>In the orbifold limit of  $K_3$  there is an additional  $SU(2)$  factor. Furthermore, for special values of the orbifold’s radii, there are up to four additional  $SU(2)$  factor, so the maximal gauge group of a  $T^2 \times K_3$  compactification is  $E_7 \times E_8 \times SU(2)^5 \times U(1)_L^2 \times U(1)_R^2$ .

down to some subgroup of  $G$  and only the gauge fields in the adjoint representation of this subgroup would remain massless. When the vacuum expectation values of the flat directions are large ( $O(M_{\text{Pl}})$ ), the massive gauge bosons and their superpartners are superheavy and hence should be integrated out of the low-energy effective theory, which would then contain only the left-over light degrees of freedom. Consequently, the flat directions responsible for the gauge symmetry breaking now reside in Abelian vector multiplets of the low-energy theory; they are a subset of the moduli fields and in this article we combine them with the dilaton field  $S$  and denote them collectively by  $\Phi^\alpha$ . If all scalars in the Cartan subalgebra of  $G$  have non-zero vacuum expectation values,  $G$  is broken down to its maximal Abelian subgroup, which can be at most  $U(1)^{22}$ ; in this case the dimension of the moduli space spanned by the vacuum expectation values of  $\Phi^\alpha$  would be  $22 + 1$  (the additional modulus corresponds to  $S$ ).

The matter fields which are charged under the gauge group are generated by primary (chiral) operators in the right-moving  $N = 4$ ,  $c = 6$  SCFT and they are members of  $N = 2$  hypermultiplets. Frequently, this right-moving SCFT also has truly marginal directions that come in  $N = 2$  hypermultiplets; they do not mix with the moduli belonging to vector multiplets and therefore span a separate, orthogonal component of the total moduli space. For example, compactifications on the  $K_3$  surface have 80 additional moduli scalars, which are gauge singlets and reside in 20  $N = 2$  hypermultiplets. Their vacuum expectation values span the moduli space of  $K_3$  [17],

$$\mathcal{M}_{K_3} = \frac{SO(20, 4)}{SO(20) \times SO(4)}. \quad (2.7)$$

However, for generic string vacua the moduli space of the hypermultiplets is unknown.

In later sections of this paper we focus on the particular subclass of four-dimensional  $N = 2$  heterotic vacua, namely compactifications of six-dimensional  $N = 1$  heterotic vacua on a two-torus  $T^2$ . The right-moving coordinates of the torus are given by the operators  $\partial X^\pm(z)$  discussed previously, but now there also exist two free complex left-moving operators  $\bar{\partial} X^\pm(\bar{z})$ , which can be used to build vertex operators for the two complex moduli of the torus  $\bar{\partial} X^\pm(\bar{z}) \partial X^\pm(z)$ . The moduli of  $T^2$  are commonly denoted by  $T = 2(\sqrt{G} + iB)$  and  $U = (\sqrt{G} - iG_{12})/G_{11}$ , where  $G_{ij}$  is the metric of  $T^2$ ,  $\sqrt{G}$  its determinant and  $B$  the constant antisymmetric-tensor background;  $U$  describes the deformations of the complex structure while  $T$  parameterizes the deformations of the area and the antisymmetric tensor, respectively. The moduli space spanned by  $T$  and  $U$  is determined by the Narain lattice of  $T^2$  [18]:

$$\mathcal{M}_{T,U} = \left( \frac{SO(2, 2)}{SO(2) \times SO(2)} \right)_{T,U} \simeq \left( \frac{SU(1, 1)}{SU(1)} \right)_T \otimes \left( \frac{SU(1, 1)}{SU(1)} \right)_U. \quad (2.8)$$

All physical properties of the two-torus compactifications are invariant under the group  $SO(2, 2, \mathbf{Z})$  of discrete duality transformations [19], which comprise the  $T \leftrightarrow U$  exchange and the  $PSL(2, \mathbf{Z})_T \times PSL(2, \mathbf{Z})_U$  dualities, which acts on  $T$  and  $U$  as

$$T \rightarrow \frac{aT - ib}{icT + d}, \quad U \rightarrow \frac{a'U - ib'}{ic'U + d'}, \quad (2.9)$$

where the parameters  $a, \dots, d'$  are integers and constrained by  $ad - bc = a'd' - b'c' = 1$ .

$T$  and  $U$  are the spin-zero components of two additional  $U(1)$   $N = 2$  vector supermultiplets. The necessary enlargement of the Abelian gauge symmetry is furnished by vertex operators of the form  $\bar{\partial}X^\pm(\bar{z}) \partial X_\mu(z)$  which generate the gauge group  $[U(1)_L]^2$ . This results in a combined gauge symmetry of  $[U(1)_+]^2 \times [U(1)_-]^2$ , where the subscript  $\pm$  indicates the combinations  $L \pm R$ . The  $[U(1)_+]^2$  originates from the internal graviton mode of the six-dimensional theory compactified on  $T^2$  while  $[U(1)_-]^2$  originates from the compactification of the six-dimensional antisymmetric tensor field. In general, the four Abelian gauge bosons transform into each other under target-space duality transformations.<sup>§</sup>

At special points in the  $(T, U)$  moduli space, additional vector fields become massless and the  $U(1)_L^2$  becomes enlarged to a non-Abelian gauge symmetry. In particular, along the critical  $T = U$  line, there are two additional massless gauge fields and the  $U(1)_L^2$  becomes  $[SU(2) \times U(1)]_L$ . The scalar superpartners of the three gauge bosons of the  $SU(2)_L$  include  $a = T - U$ , which acts as the Higgs field breaking the  $SU(2)_L$  when one moves away from the  $T = U$  line [20]. Similar critical lines exist for  $T \equiv U \pmod{SL(2, \mathbf{Z})}$ , *i. e.*,  $T = (aU - ib)/(icU + d)$  for some integer  $a, b, c, d$  with  $ad - bc = 1$ . When two such lines intersect, each line brings with it a pair of massless gauge fields and the gauge symmetry becomes enhanced even further; the enhanced group may be determined by simply counting the intersecting critical lines [21]. For example, the point  $T = U = 1$  lies at the intersection of two critical lines, namely  $T = U$  and  $T = 1/U$ , and hence has four extra gauge bosons. The corresponding gauge symmetry is  $SU(2)_L^2$  and the two Higgs fields in the Cartan subalgebra of this symmetry can be identified as  $a_1 = T - U$  and  $a_2 = T - (1/U)$ <sup>¶</sup>. Similar two-line intersections happen whenever  $T \equiv U \equiv 1 \pmod{SL(2, \mathbf{Z})}$  and the gauge group is enhanced to an  $SU(2)_L^2$  at all such points. On the other hand, three critical lines  $T = U$ ,  $T = 1/(U - i)$  and  $T = (iU + 1)/U$  intersect at the critical point  $T = U = \rho = e^{2\pi i/12}$ , where one therefore has six massless gauge bosons in addition to the  $U(1)_L^2$ ; this enhances the gauge symmetry all the way to an  $SU(3)_L$ . Two Higgs scalars in its Cartan subalgebra of this symmetry can be identified as *e. g.*,  $a_1 = T - U$  and  $a_2 = ((iT + 1)/T) - (1/(U - i))$ . Again, similar triple intersections occur at  $T \equiv U \equiv \rho \pmod{SL(2, \mathbf{Z})}$  and the gauge group is enhanced to an  $SU(3)_L$  at all such points. The above is the complete list of all the critical lines and points of the  $(T, U)$  moduli space; at all the other point, the  $(T, U)$  system has only the  $U(1)_L^2$  gauge symmetry. In particular, there is no enlargement of the gauge symmetry

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<sup>§</sup>For example, the transformation  $T \rightarrow 1/T$  rotates the left- and right-moving torus coordinates since it involves an exchange of momentum and winding numbers (see [20] for details). This can be seen as a transformation on the world-sheet electric and magnetic charges. In particular, for the simple case of  $U = 1$ ,  $\text{Im } T = 0$ , the transformation  $T \rightarrow 1/T$  just acts as  $\partial X^\pm(z) \rightarrow -\partial X^\pm(z)$  with  $\bar{\partial}X^\pm(\bar{z})$  invariant and hence the groups  $[U(1)_+]^2$  and  $[U(1)_-]^2$  are simply interchanged.

<sup>¶</sup>Alternatively, the Higgs fields can be defined as  $a'_1 = \frac{T-1}{T+1}$ ,  $a'_2 = \frac{U-1}{U+1}$ ;  $(\frac{T-1}{T+1})^2$  and  $(\frac{U-1}{U+1})^2$  correspond to the uniformizing variables of modular functions around the critical points  $T = 1$  and  $U = 1$ . For the case an enhanced  $SU(3)_L$  gauge group the analogous definitions are  $a'_1 = \frac{T-\rho}{T+\rho}$  and  $a'_2 = \frac{U-\rho}{U+\rho}$ .



when  $T \equiv 1$  but  $U \not\equiv 1$  or  $T \equiv \rho$  but  $U \not\equiv \rho$ ; this fact will be important for our analysis in section 4.3.

At the critical points, complete  $N = 2$  supermultiplets become massless and the vertex operators of their bosonic components  $(A_\mu, C^\pm)$  have the form

$$\begin{aligned} A_\mu &\sim e^{i(\bar{p}^+(T,U)\bar{X}^-(\bar{z})+\bar{p}^-(T,U)\bar{X}^+(\bar{z}))} e^{i(p^+(T,U)X^-(z)+p^-(T,U)X^+(z))} \partial X_\mu(z), \\ C^\pm &\sim e^{i(\bar{p}^+(T,U)\bar{X}^-(\bar{z})+\bar{p}^-(T,U)\bar{X}^+(\bar{z}))} e^{i(p^+(T,U)X^-(z)+p^-(T,U)X^+(z))} \partial X^\pm(z), \end{aligned} \quad (2.10)$$

where the Narain lattice vectors satisfy

$$\bar{p}^+(T,U)\bar{p}^-(T,U) = p^+(T,U)p^-(T,U) + 2. \quad (2.11)$$

The masses of such states are given by  $m^2(T,U) = \frac{1}{2}p^+(T,U)p^-(T,U)$ , which indeed vanish precisely at the critical points. Away from the critical points, the multiplet  $(A_\mu, C^\pm)$  has non-vanishing right-moving lattice momentum vectors  $(p^+, p^-)$ , which implies that these massive states have non-vanishing central charges. Therefore they build small representations of the  $N = 2$  supersymmetry algebra.

Toroidal compactifications can be continuously deformed by turning on non-trivial Wilson lines in the gauge group for the two periods of the two-torus. Such deformations give rise to additional moduli belonging to Abelian vector supermultiplets; we denote such Wilson-line moduli by  $\phi^i$  with  $i = 1, \dots, P$  ( $P \leq 20$ ). The combined moduli space spanned by  $T, U$  and  $\phi^i$  can be directly derived from the Narain lattice of the heterotic string compactification on  $T^2$  [18, 22]. One finds the symmetric Kähler space (see also section 4):

$$\mathcal{M}_{T,U,\phi^i} = \left( \frac{SO(2, P+2)}{SO(2) \times SO(P+2)} \right) / SO(2, 2+P, \mathbf{Z}). \quad (2.12)$$

(see also section 4). Together with the dilaton field  $S$ , which parameterizes the coset space  $SU(1,1)/U(1)$ , we are thus dealing with an  $(3+P)$ -dimensional space, spanned by the complex moduli  $\Phi^\alpha$ . At a *generic* point in this moduli space the gauge group is  $U(1)_{L+R}^2 \times U(1)_{L-R}^2 \times U(1)^P$ ; it is enlarged to a non-Abelian gauge group at special points (or rather subspaces) of the moduli space.

Target-space duality transformations now act simultaneously on all moduli fields  $T, U$  and the additional moduli  $\phi^i, i = 1, \dots, P$ . Specifically, the target-space duality transformations are contained in the discrete group  $SO(2, 2+n, \mathbf{Z})$ , which possesses  $PSL(2, \mathbf{Z})_T \times PSL(2, \mathbf{Z})_U \subset SO(2, 2, \mathbf{Z})$  as a subgroup. For example,  $PSL(2, \mathbf{Z})_T$  acts on  $T$  in the standard way (see eq.(2.9)); the  $\phi^i$  transform with modular weight  $-1$  under this transformation, *i.e.*,

$$\phi^i \rightarrow \frac{\phi^i}{icT + d}. \quad (2.13)$$

However  $U$  transforms also non-trivially under this transformation as [23, 22]:

$$U \rightarrow U - \frac{ic}{icT + d} \phi^i \phi^i. \quad (2.14)$$

Thus, in the presence of the  $\phi^i$ ,  $T$  and  $U$  get mixed under duality transformations, which is a reflection of the non-factorizable structure of the moduli space.

# 3 Effective Quantum Field Theories

## with Local $N = 2$ Supersymmetry

In this section we summarize generic properties of the effective  $N = 2$  supergravity action with particular emphasis on the couplings of the vector multiplets. The section is divided into two parts; in 3.1 we summarize and further develop a number of useful results of special geometry and in 3.2 we discuss quantum effects in effective  $N = 2$  supersymmetric theories.

### 3.1 Summary of special geometry

In  $N = 2$  supersymmetric Yang-Mills theory the action is encoded in a holomorphic prepotential  $F(X)$ , where  $X^A$  ( $A = 1, \dots, n$ ) denote the vector superfields and also the complex scalar components of such superfields. The function  $F(X)$  is usually assumed to be invariant under the gauge group, although this requirement is not always necessary [24, 25]. Two different functions  $F(X)$  may correspond to equivalent equations of motion; generically the equivalence involves symplectic reparametrizations combined with duality transformations, which we will turn to shortly. The local  $N = 2$  supersymmetry requires an additional vector superfield  $X^0$  in order to accommodate the graviphoton, but the scalar and the spinor components of this superfield do not lead to additional physical particles. Therefore, in the local case  $F(X)$  is a holomorphic function of  $n + 1$  complex variables  $X^I$  ( $I = 0, 1, \dots, n$ ), but it must be a *homogeneous* function of degree two [9]. According to the superconformal multiplet calculus, the physical scalar fields of this system parameterize an  $n$ -dimensional complex hypersurface, defined by the condition that the imaginary part of  $X^I \bar{F}_I(\bar{X})$  must be a constant linearly related to Planck's constant, while the overall phase of the  $X^I$  is irrelevant in view of a local chiral invariance. The embedding of this hypersurface can be described in terms of  $n$  complex coordinates  $z^A$  by letting  $X^I$  be proportional to some holomorphic sections  $X^I(z)$  of the projective space. The resulting geometry for the space of physical scalar fields belonging to vector multiplets of an  $N = 2$  supergravity is a *special* Kähler geometry [9, 26], with a Kähler metric  $g_{AB} = \partial_A \partial_{\bar{B}} K(z, \bar{z})$  following from a Kähler potential of the special form\*

$$K(z, \bar{z}) = -\log \left( i \bar{X}^I(\bar{z}) F_I(X(z)) - i X^I(z) \bar{F}_I(\bar{X}(\bar{z})) \right). \quad (3.1)$$

The curvature tensor associated with such a special Kähler space satisfies the characteristic relation [27]

$$R^A_{BC}{}^D = 2\delta^A_{(B} \delta^D_{C)} - e^{2K} \mathcal{W}_{BCE} \bar{\mathcal{W}}^{EAD}, \quad (3.2)$$

where

$$\mathcal{W}_{ABC} = F_{IJK}(X(z)) \frac{\partial X^I(z)}{\partial z^A} \frac{\partial X^J(z)}{\partial z^B} \frac{\partial X^K(z)}{\partial z^C}. \quad (3.3)$$

---

\*Here and henceforth we use the standard convention where  $F_{IJ\dots}$  denote multiple derivatives with respect to  $X$  of the holomorphic prepotential.

Up to an irrelevant phase, the proportionality factor between the  $X^I$  and the holomorphic sections  $X^I(z)$  is equal to  $\exp(\frac{1}{2}K(z, \bar{z}))$ . A convenient choice of inhomogeneous coordinates  $z^A$  are the *special* coordinates, defined by

$$z^A = X^A/X^0, \quad A = 1, \dots, n, \quad (3.4)$$

or, equivalently,

$$X^0(z) = 1, \quad X^A(z) = z^A. \quad (3.5)$$

In this parameterization the Kähler potential can be written as [28]

$$K(z, \bar{z}) = -\log \left( 2(\mathcal{F} + \bar{\mathcal{F}}) - (z^A - \bar{z}^A)(\mathcal{F}_A - \bar{\mathcal{F}}_A) \right), \quad (3.6)$$

where  $\mathcal{F}(z) = i(X^0)^{-2}F(X)$ .

The Lagrangian terms containing the kinetic energies of the gauge fields are

$$\mathcal{L}^{\text{gauge}} = -\frac{i}{8} \left( \mathcal{N}_{IJ} F_{\mu\nu}^{+I} F^{+\mu\nu J} - \bar{\mathcal{N}}_{IJ} F_{\mu\nu}^{-I} F^{-\mu\nu J} \right), \quad (3.7)$$

where  $F_{\mu\nu}^{\pm I}$  denote the selfdual and anti-selfdual field-strength components and

$$\mathcal{N}_{IJ} = \bar{F}_{IJ} + 2i \frac{\text{Im}(F_{IK}) \text{Im}(F_{JL}) X^K X^L}{\text{Im}(F_{KL}) X^K X^L}. \quad (3.8)$$

Hence  $\mathcal{N}$  is the field-dependent tensor that comprises the inverse gauge couplings  $g_{IJ}^{-2} = \frac{i}{4}(\mathcal{N}_{IJ} - \bar{\mathcal{N}}_{IJ})$  and the generalized  $\theta$  angles  $\theta_{IJ} = 2\pi^2(\mathcal{N}_{IJ} + \bar{\mathcal{N}}_{IJ})$ . Note the important identity  $F_I = \mathcal{N}_{IJ} X^J$ .

As we already mentioned, different functions  $F(X)$  can lead to equivalent equations of motion. Such equivalence often involves the electric-magnetic duality of the field strengths rather than local transformations of the vector potentials  $A_\mu^I$ . For the non-Abelian case such a duality does not make sense (because the field equations depend explicitly on the vector potentials), but it is perfectly legitimate in the context of Abelian gauge fields when all the *fundamental* fields of the theory are neutral.<sup>†</sup> With this proviso in mind, let us introduce the duality transformations. Following ref. [9] and appendix C of ref. [29]<sup>‡</sup>, we define the tensors  $G_{\mu\nu I}^\pm$  as

$$G_{\mu\nu I}^+ = \mathcal{N}_{IJ} F_{\mu\nu}^{+J}, \quad G_{\mu\nu I}^- = \bar{\mathcal{N}}_{IJ} F_{\mu\nu}^{-J}. \quad (3.9)$$

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<sup>†</sup>A local fundamental field can be electrically charged but it cannot carry a magnetic charge. On the other hand, an extended object like a soliton can have both electric and magnetic charges. Therefore, when all the fundamental fields are neutral, one is free to choose any integral basis for the electric and magnetic charges, but a charged local field (in particular, a non-Abelian gauge field) restricts this choice since its charge must be electric rather than magnetic.

<sup>‡</sup>As compared to the definitions in [9, 27, 29], our notation is as follows:

$$\begin{aligned} [K(z, \bar{z})]^\text{here} &= -K(z, \bar{z}) - \log 2, & [\mathcal{W}_{ABC}]^\text{here} &= -2iQ_{ABC}. \\ [F(X)]^\text{here} &= -\frac{i}{2}F(X), & [G_{\mu\nu I}^+]^\text{here} &= -iG_{\mu\nu I}^+, & [\mathcal{N}_{IJ}(X, \bar{X})]^\text{here} &= 2i\mathcal{N}_{IJ}(X, \bar{X}), \end{aligned}$$

Note that the change in the Kähler potential induces a change of sign in the Kähler metric.

Then the set of Bianchi identities and equations of motion for the Abelian gauge fields can be written as

$$\partial^\mu (F_{\mu\nu}^{+I} - F_{\mu\nu}^{-I}) = 0, \quad \partial^\mu (G_{\mu\nu I}^+ - G_{\mu\nu I}^-) = 0, \quad (3.10)$$

which are invariant under the transformations

$$\begin{aligned} F_{\mu\nu}^{+I} &\longrightarrow \tilde{F}_{\mu\nu}^{+I} = U^I{}_J F_{\mu\nu}^{+J} + Z^{IJ} G_{\mu\nu J}^+, \\ G_{\mu\nu I}^+ &\longrightarrow \tilde{G}_{\mu\nu I}^+ = V_I{}^J G_{\mu\nu J}^+ + W_{IJ} F_{\mu\nu}^{+J}, \end{aligned} \quad (3.11)$$

where  $U$ ,  $V$ ,  $W$  and  $Z$  are constant, real,  $(n+1) \times (n+1)$  matrices. The transformations for the anti-selfdual tensors follow by complex conjugation. However, to ensure that (3.9) remains satisfied with a symmetric tensor  $\mathcal{N}$ , at least in the generic case, the transformation (3.11) must be symplectic (disregarding an overall multiplication of the field strength tensors by a real constant). More precisely,

$$\mathcal{O} \stackrel{\text{def}}{=} \begin{pmatrix} U & Z \\ W & V \end{pmatrix} \quad (3.12)$$

must be an  $Sp(2n+2, \mathbf{R})$  symplectic matrix, that is, it must satisfy

$$\mathcal{O}^{-1} = \Omega \mathcal{O}^T \Omega^{-1} \quad \text{where} \quad \Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (3.13)$$

For the sub-matrices  $U$ ,  $V$ ,  $W$  and  $Z$ , this means

$$\begin{aligned} U^T V - W^T Z &= V^T U - Z^T W = \mathbf{1}, \\ U^T W &= W^T U, \quad Z^T V = V^T Z. \end{aligned} \quad (3.14)$$

Incidentally, it follows straightforwardly from (3.11) that the kinetic term of the vector fields (3.10) does not generically preserve its form under  $Sp(2n+2, \mathbf{R})$  because

$$\begin{aligned} \tilde{F}_{\mu\nu}^{+I} \tilde{G}_I^{+\mu\nu} &= F_{\mu\nu}^{+I} G_I^{+\mu\nu} + (U^T W)_{IJ} F_{\mu\nu}^{+I} F^{+\mu\nu J} \\ &\quad + 2(W^T Z)_I{}^J F_{\mu\nu}^{+I} G_J^{+\mu\nu} + (Z^T V)^{IJ} G_{\mu\nu I}^+ G_J^{+\mu\nu}, \end{aligned} \quad (3.15)$$

which confirms that generally it is only the combined equations of motion and Bianchi identities that are equivalent, but not the Lagrangian or the action.

Next, consider the transformation rules for the scalar fields.  $N = 2$  supersymmetry relates the  $X^I$  to the field strengths  $F_{\mu\nu}^{+I}$ , while the  $F_I$  are related to the  $G_I^{+\mu\nu}$ . Hence, eqs. (3.11) suggest

$$\begin{aligned} \tilde{X}^I &= U^I{}_J X^J + Z^{IJ} F_J, \\ \tilde{F}_I &= V_I{}^J F_J + W_{IJ} X^J, \end{aligned} \quad (3.16)$$

Owing to the symplectic conditions (3.13), the quantities  $\tilde{F}_I$  can be written as the derivative of a new function  $\tilde{F}(\tilde{X})$  with respect to the new coordinate  $\tilde{X}^I$ :

$$\tilde{F}(\tilde{X}) = \frac{1}{2}(U^T W)_{IJ} X^I X^J + \frac{1}{2}(U^T V + W^T Z)_I{}^J X^I F_J + \frac{1}{2}(Z^T V)^{IJ} F_I F_J, \quad (3.17)$$

where we made use of the homogeneity of  $F$ .<sup>§</sup> Lagrangians parameterized by  $F(X)$  and  $\tilde{F}(\tilde{X})$  represent equivalent theories, at least for the Abelian sector of the theory. Furthermore, when

$$\tilde{F}(\tilde{X}) = F(\tilde{X}), \quad (3.18)$$

the field equations are *invariant* under the symplectic transformations. Note that this does *not* imply that  $F$  itself is an invariant function in the usual sense. Indeed, from comparison to (3.17) one readily verifies that  $F(\tilde{X}) \neq F(X)$ , as was already observed in [9] for infinitesimal transformations. A consequence of (3.18) is that substituting  $\tilde{X}$  for  $X$  in  $F_I(X)$  induces precisely the symplectic transformation specified in the second formula of (3.16). In practice this is a more direct way to verify the invariance, rather than checking (3.18).

Let us now present some additional details on the generic transformation rules for various tensors. In view of recent interest in duality transformations for rigidly supersymmetric Yang-Mills theories, we stress that most of these results apply both to local and rigid  $N = 2$  supersymmetry. We have seen that  $(X^I, F_J)$  and  $(F_{\mu\nu}^+, G_{\mu\nu J}^+)$  transform linearly as  $(2n + 2)$ -component vectors under  $Sp(2n + 2, \mathbf{R})$ . Defining

$$\frac{\partial \tilde{X}^I}{\partial X^J} \equiv \mathcal{S}^I{}_J(X) = U^I{}_J + Z^{IK} F_{KJ}, \quad (3.19)$$

we note that the second derivative of  $F(X)$  changes as a period matrix under  $Sp(2n + 2, \mathbf{R})$ ,

$$\tilde{F}_{IJ} = (V_I^K F_{KL} + W_{IL}) [\mathcal{S}^{-1}]^L{}_J, \quad (3.20)$$

where the right-hand side depends on the coordinates  $X^I$  through  $F_{IJ}$  and  $\mathcal{S}$ . For this reason  $(X^I, F_J)$  are called the periods; in string theory they correspond to the periods of certain harmonic forms and in rigidly supersymmetric Yang-Mills theory they are also very useful for understanding the non-perturbative features of the theory [3]. Actually, the periods are more fundamental than the underlying function  $F(X)$ , because there are situations where the transformations  $\tilde{X}(X)$  are singular, so that a meaningful holomorphic prepotential cannot be found; this was demonstrated recently in [10]. Although in this case it is still possible to define a homogeneous function  $\tilde{F}(\tilde{X}) = \frac{1}{2} \tilde{X}^I \tilde{F}_I$ , this function does not depend on all the coordinates  $\tilde{X}^I$ . Often, such a  $\tilde{F}(\tilde{X})$  vanishes identically, but this is not generic. In spite of the non-existent prepotential, however, the underlying theory is well defined and can be related via symplectic reparameterizations to theories where a prepotential does exist [10]. Such situations are indicated by a singular matrix of second derivatives  $\tilde{F}_{IJ}$  (cf. (3.20)) while the matrix  $\tilde{\mathcal{N}}_{IJ}$  of inverse gauge couplings remains well-behaved (cf. (3.22) below). In the purely Abelian case, one can always choose a coordinate basis  $X^I$  for which the prepotential  $F$  does exist, but this may fail in the presence of a non-Abelian gauge group where electric-magnetic dualities are not legitimate. In section 4, we shall see that in string theory, the uniform dilaton dependence of the gauge couplings is manifest only in a basis where  $F$  does not exist, although

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<sup>§</sup>For rigid  $N = 2$  supersymmetry where  $F$  is not homogeneous, one should add  $F(X) - \frac{1}{2} X^I F_I(X)$  to the right-hand side of (3.17) and one may also add terms constant or linear in  $\tilde{X}^I$ .

this is not an obstacle for considering the loop corrections to the prepotential defined for another basis. Here, we would like to add a comment that in situations without a prepotential, the appropriate criterion for invariance of the equations of motion is not eq. (3.18) (which is not quite meaningful in this case), but whether the transformation of the periods can be correctly induced (up to an overall holomorphic proportionality factor) by appropriate changes of the underlying coordinates  $z^A$ .

The transformation rules for the tensors  $\text{Im}(F_{IJ})$  and  $F_{IJK}$  are as follows:

$$\begin{aligned}\text{Im}(\tilde{F}_{IJ}) &= \text{Im}(F_{KL}) [\bar{\mathcal{S}}^{-1}]^K_I [\mathcal{S}^{-1}]^L_J \\ \tilde{F}_{IJK} &= F_{MNP} [\mathcal{S}^{-1}]^M_I [\mathcal{S}^{-1}]^N_J [\mathcal{S}^{-1}]^P_K.\end{aligned}\tag{3.21}$$

The transformation rules for the gauginos are also given by the  $\mathcal{S}^{-1}$  or its complex conjugate, depending on chirality. In the rigidly supersymmetric case, the transformation of the gauge field strengths  $F_{\mu\nu}^{\pm I}$  is also described by this matrix, but in the locally supersymmetric case this transformation is modified as the tensor  $\mathcal{N}_{IJ}$  governing the relation between the  $F_{\mu\nu}^{\pm I}$  and the  $G_{\mu\nu}^{\pm I}$  is no longer equal to  $\frac{1}{4}\bar{F}_{IJ}$  or even anti-holomorphic. Nevertheless, the transformation rule for the tensor  $\mathcal{N}$  itself is precisely as in the rigid case, namely

$$\tilde{\mathcal{N}}_{IJ} = (V_I^K \mathcal{N}_{KL} + W_{IL}) [(U + Z\mathcal{N})^{-1}]^L_J.\tag{3.22}$$

To obtain this last result it is crucial that the function  $F(X)$  is homogeneous of second degree. Note that the symmetry of  $\tilde{F}_{IJ}$  and  $\tilde{\mathcal{N}}_{IJ}$  is ensured by the symplectic conditions (3.13).

Three particular subgroups of the  $Sp(2n+2, \mathbf{R})$  will be relevant to our discussion in section 4. The first subgroup contains the classical target-space duality transformations which are symmetries of the tree-level Lagrangian. From eqs. (3.15), (3.16) we learn that the Lagrangian is left invariant by the subgroup that satisfies  $W = Z = 0$  and  $V^T = U^{-1}$ . For the second subgroup, we continue to demand  $Z = 0$  but relax the  $W = 0$  condition; according to eq. (3.14), we then should have  $V^T = U^{-1}$  and  $W^T U$  should be a symmetric matrix. These conditions lead to *semiclassical* transformations of the form

$$\begin{aligned}\tilde{X}^I &= U^I_J X^J, & \tilde{F}_I &= [U^{-1}]^J_I F_J + W_{IJ} X^J, \\ \tilde{F}_{\mu\nu}^{\pm I} &= U^I_J F_{\mu\nu}^{\pm J}, & \tilde{\mathcal{N}} &= [U^{-1}]^T \mathcal{N} U^{-1} + W U^{-1},\end{aligned}\tag{3.23}$$

which can always be implemented as Lagrangian symmetries of the vector fields  $A_\mu^I$ . The last term in the last equation in (3.23) amounts to a constant shift of the theta angles; at the quantum level, such shifts are quantized and hence the symplectic group must be restricted to  $Sp(2n+2, \mathbf{Z})$ . We will see that such shifts in the  $\theta$ -angle do occur whenever the one-loop gauge couplings have logarithmic singularities at special points in the moduli space where massive modes become massless. Therefore, these symmetries are related to the semi-classical (one-loop) monodromies around such singular points. The third subgroup contains elements that interchange the field-strength tensors  $F_{\mu\nu}^I$  and  $G_{\mu\nu I}$  and correspond to electric-magnetic dualities. These transformations are defined

by  $U = V = 0$  and  $W^T = -Z^{-1}$ , which yields

$$\tilde{\mathcal{N}} = -W \mathcal{N}^{-1} W^T, \quad (3.24)$$

so that they give rise to an inversion of the gauge couplings and hence must be non-perturbative in nature. In the heterotic string context, such transformations are often called  $S$ -dualities because of the way they act upon the dilaton field  $S$ . We shall return to this issue in section 4.1.

Let us now turn our attention to the holomorphic prepotentials of the following special form

$$F(X) = \frac{1}{2} d_{ABC} \frac{X^A X^B X^C}{X^0}, \quad (3.25)$$

where  $d_{ABC}$  are some real constants. The theories described by this class of  $F$ -functions emerge via dimensional reduction from five-dimensional  $N = 2$  supergravity coupled to vector multiplets; they also emerge in the heterotic string context, regardless of any dimensional reduction. Let us therefore record a few convenient formulae for future use. In special coordinates,

$$\begin{aligned} \mathcal{W}_{ABC} &= 3d_{ABC}, \\ \text{Im}(F_{AK})X^K(z) &= -\frac{3i}{4}d_{ABC}(z - \bar{z})^B(z - \bar{z})^C, \\ \text{Im}(F_{0K})X^K(z) &= \frac{i}{4}d_{ABC}(z^A z^B z^C - 3\bar{z}^A \bar{z}^B z^C + 2\bar{z}^A \bar{z}^B \bar{z}^C), \\ \text{Im}(F_{KL})X^K(z)\bar{X}^L(\bar{z}) &= -\frac{1}{2}e^{-K(z, \bar{z})}, \\ \text{Im}(F_{KL})X^K(z)X^L(z) &= e^{-K(z, \bar{z})}, \end{aligned} \quad (3.26)$$

while the Kähler potential is given by

$$K(z, \bar{z}) = -\log \left( -\frac{1}{2} i d_{ABC} (z - \bar{z})^A (z - \bar{z})^B (z - \bar{z})^C \right). \quad (3.27)$$

The special Kähler spaces corresponding to (3.25) always possess continuous isometries [27], which in special coordinates take the form [29]

$$\delta z^A = b^A - \frac{2}{3}\beta z^A + \tilde{B}^A_B z^B - \frac{1}{2}(R^A_{BC}{}^D a_D) z^B z^C. \quad (3.28)$$

Here  $\beta$ ,  $b^A$  and  $a_A$  are real parameters and the matrix  $\tilde{B}$  parameterizes the infinitesimal real linear transformations of the  $X^A$  under which  $d_{ABC} X^A X^B X^C$  is left invariant; the isometries corresponding to the parameters  $a_A$  exist only for those parameters for which the  $R^A_{BC}{}^D a_D$  are constant. All homogeneous spaces of this type have been classified and it has been shown that all their isometries are related to the symplectic transformations discussed above [27, 29]. The infinitesimal form of the matrices  $U$ ,  $V$ ,  $W$  and  $Z$  was first determined in [27]. Introducing the notation

$$\mathcal{O} = \mathbf{1} + \begin{pmatrix} B & -D \\ C & -B^T \end{pmatrix}, \quad (3.29)$$

we have (the first row and column refer to the  $I = 0$  component)

$$B_J^I = \begin{pmatrix} \beta & a_B \\ b^A & \tilde{B}_B^A + \frac{1}{3}\beta \delta_B^A \end{pmatrix}, \quad C_{IJ} = \begin{pmatrix} 0 & 0 \\ 0 & 3(db)_{AB} \end{pmatrix}, \quad D^{IJ} = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{4}{9}(C a)^{AB} \end{pmatrix}. \quad (3.30)$$

where we use an obvious notation where  $(db)_{AB} = d_{ABC}b^C$ ,  $(dbb)_A = d_{ABC}b^Bb^C$ ,  $(dbbb) = d_{ABC}b^Ab^Bb^C$ , and likewise for contractions of  $C^{ABC}$  with the parameters  $a_A$ , where  $C^{ABC}$  is defined by

$$C^{ABC} = -\frac{9}{4}e^{2K(z,\bar{z})}\bar{\mathcal{W}}^{ABC}(z,\bar{z}). \quad (3.31)$$

Obviously, in general  $C^{ABC}$  are not constant, but again the possible parameters  $a_A$  are restricted by the condition that  $C^{ABC}a_C$  should be constant. For homogeneous spaces, there are always nontrivial solutions for the  $a_A$ , while for symmetric spaces  $C^{ABC}$  are constant and so there are precisely  $n$  isometries associated with  $n$  independent parameters  $a_A$ . All homogeneous spaces corresponding to the functions (3.25) have been classified in [29].

In string theory the continuous isometries are not preserved by world-sheet instanton effects. Therefore infinitesimal isometries and corresponding duality transformations are only relevant for certain couplings. In that context we observe that the isometries associated with the parameters  $b^A$  and  $a_A$  can simply be exponentiated to finite symplectic transformations, which for special values of  $b^A$  and  $a_A$  are exact symmetries of the underlying string theory. Of course, these symmetries can also be determined from string arguments alone and it will be instructive to compare the results. For the symplectic transformation associated with finite parameters  $b^A$ , we have

$$U(b) = V^T(-b) = \begin{pmatrix} 1 & 0 \\ b^A & \mathbf{1}_n \end{pmatrix}, \quad W(b) = \frac{1}{2} \begin{pmatrix} -(dbbb) & -3(db b)_B \\ 3(db b)_A & 6(db)_{AB} \end{pmatrix}, \quad Z(b) = 0, \quad (3.32)$$

with the corresponding transformations of the special coordinates being simply

$$z^A \longrightarrow z^A + b^A. \quad (3.33)$$

These transformations are of type (3.23) and thus can be realized on the vector potentials and leave the Lagrangian invariant up to a total divergence corresponding to a shift in the  $\theta$  angles.<sup>¶</sup> For specific choices of the  $d_{ABC}$ , the above results can now be compared to those derived directly from string theory [30, 31, 32].

Likewise, the symplectic transformations associated with finite parameters  $a_A$  are

$$U(a) = V^T(-a) = \begin{pmatrix} 1 & a_B \\ 0 & \mathbf{1}_n \end{pmatrix}, \quad W(a) = 0, \quad Z(a) = \frac{2}{27} \begin{pmatrix} (Caaa) & 3(Caa)^B \\ -3(Caa)^A & 6(Ca)^{AB} \end{pmatrix}, \quad (3.34)$$

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<sup>¶</sup>We should stress that this result holds in the basis associated with (3.25). As we shall exhibit in subsection 4.1, in another symplectic basis the situation can be qualitatively different.



and the corresponding transformations on special coordinates take the form

$$z^A \longrightarrow \frac{z^A + \frac{2}{3}(C a)^{AB}(dzz)_B + \frac{1}{9}(C aa)^A(dzzz)}{1 + a_B z^B + \frac{1}{3}(C aa)^B(dzz)_B - \frac{1}{27}(C aaa)(dzzz)} . \quad (3.35)$$

For appropriate values of the parameters the matrices (3.32) and (3.34) may generate the group of discrete transformations that are preserved at the quantum level. For the symmetric Kähler space relevant for the heterotic string compactifications, the  $SL(2, \mathbf{Z})$  groups associated with target space- and  $S$ -dualities can be generated in this way, as their nilpotent subgroups are special cases of (3.32) and (3.34). We return to this in section 4.1.

### 3.2 Quantum effects in $N = 2$ theories

Classically, the geometry of the field space is unrelated to the field dependence of the particles' masses. However, an effective quantum field theory (EQFT) has to be cut-off at the Planck scale and thus should not include any of the superheavy states. The distinction between the light fields that should be manifest in the low-energy EQFT and the heavy fields that should be integrated out depends on the moduli parameters of the underlying string vacuum. In general, a single connected family of string vacua gives rise to several distinct low-energy EQFTs according to the moduli-dependent spectra of the light particles. Therefore, from any particular EQFT's point of view, there is a difference between the spectrum-preserving moduli scalars, whose vacuum expectation values may become arbitrarily large without giving a Planck-sized mass to any otherwise light particles, and between all the other flat directions of the scalar potential. Physically, the latter may also develop arbitrarily large vacuum expectation values, but in that case the spectrum of the light particles would no longer agree with the original EQFT and one has to switch over to a different EQFT in order to properly describe the low-energy limit of the string vacuum. Consequently, the field-dependent couplings of the EQFT should be written as complete analytic functions of the spectrum-preserving moduli fields, but their dependence on all the other field may be described by a truncated power series. (See ref. [14] for a more detailed discussion.)

In this spirit, we divide the scalars  $z^A = X^A/X^0$  belonging to vector multiplets of an  $N = 2$  locally supersymmetric EQFT into the spectrum-preserving moduli  $\Phi^\alpha = -iz^\alpha$  and the “matter” scalars  $C^a = -iz^{a*}$  and expand the prepotential  $\mathcal{F}$  of the theory as a truncated power series in the latter:

$$\mathcal{F}(\Phi, C) = h(\Phi) + \sum_{ab} f_{ab}(\Phi) C^a C^b + \dots \quad (3.36)$$

Obviously, all scalars in the non-Abelian vector multiplets should be regarded as matter (there are flat directions among these scalars, but none are spectrum preserving since

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\*The  $-i$  is included in order to be consistent with the standard string conventions. For toroidal compactifications,  $\Phi^\alpha$  correspond to the  $3 + P$  complex moduli fields  $S, T, U$  and  $\phi^i$  introduced in section 2.

their vacuum expectation values induce a mass for some of the non-Abelian fields); for such non-Abelian matter, the gauge symmetry of the prepotential requires

$$f_{ab}(\Phi) = \delta_{ab} f_{(a)}(\Phi) \quad (3.37)$$

where the index  $(a)$  refers to the appropriate irreducible factor  $G_{(a)}$  of the gauge group  $G = \prod_{(a)} G_{(a)}$ . Similarly, if any hypermultiplets appearing in the EQFT are charged under an Abelian gauge symmetry, the scalar superpartner of that gauge boson should be regarded as matter since its vacuum expectation value would give masses to all such charged hypermultiplets. On the other hand, if all the *light* particles are neutral with respect to some Abelian gauge field, then its scalar superpartner is a spectrum-preserving modulus. To be precise, we divide the Abelian vector multiplets into  $\Phi^\alpha$  and  $C^a$  such that all the light hypermultiplets of the EQFT under consideration are exactly massless for  $C^a = 0$  and arbitrary  $\Phi^\alpha$ .

A proper discussion of the field-dependent couplings of an effective *quantum* field theory must distinguish between two kinds of renormalized couplings [11]: First, there are *effective couplings* associated with physical processes; for example, Coulomb-like scattering of charged particles defines a momentum-dependent gauge coupling  $g(p^2)$ . The momentum dependence of such couplings is unavoidable in theories with massless charged particles; therefore, the effective couplings generally cannot be summarized in any *local* effective Lagrangian. Second, there are the *Wilsonian couplings*, which are the coefficients of the quantum operator products in the action functional of the theory. Similar to its classical counterpart, this Wilsonian action is an  $\int d^4x$  of a local Wilsonian Lagrangian; consequently, the supersymmetric constraints satisfied by the Wilsonian couplings of an EQFT are the same as in the classical case.<sup>†</sup> In particular, the Wilsonian prepotential of an  $N = 2$  supersymmetric EQFT must be a holomorphic function  $\mathcal{F}(\Phi, C)$  defining the Wilsonian Kähler function  $K(z, \bar{z})$  according to eq. (3.6). In light of the expansion (3.36), we have

$$\begin{aligned} K(\Phi, \bar{\Phi}, C, \bar{C}) &= K_\Phi(\Phi, \bar{\Phi}) + \sum_{ab} Z_{ab}(\Phi, \bar{\Phi}) C^a \bar{C}^b + \dots, \\ \text{where } K_\Phi(\Phi, \bar{\Phi}) &= -\log \left[ 2(h + \bar{h}) - \sum_\alpha (\Phi^\alpha + \bar{\Phi}^\alpha)(\partial_\alpha h + \bar{\partial}_\alpha h) \right], \\ \text{and } Z_{ab}(\Phi, \bar{\Phi}) &= 4e^{K_\Phi(\Phi, \bar{\Phi})} \text{Re } f_{ab}(\Phi). \end{aligned} \quad (3.38)$$

Similarly, the Wilsonian gauge couplings follow from eq. (3.8). Since the distinction between the matter scalars  $C^a$  and the spectrum-preserving moduli  $\Phi^\alpha$  (henceforth called simply moduli) presumes  $|C^a| \ll 1$  (in Planck units), it follows that the vector superpartners of the  $C^a$  do not mix with the graviphoton and hence the corresponding Wilsonian gauge couplings are simply  $(g_{ab}^{-2})^W = \text{Re } f_{ab}(\Phi)$ . In particular, for the non-Abelian gauge

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<sup>†</sup>This presumes that the quantum theory is regularized in a way that preserves both the local supersymmetry and the four-dimensional background gauge invariance. See ref. [12] for the discussion of these issues in the local  $N = 1$  case.

fields  $(g_{(a)}^{-2})^W = \text{Re } f_{(a)}(\Phi)$ . On the other hand, the vector superpartners of the moduli  $\Phi^\alpha$  do mix with the graviphoton; consequently, the corresponding Wilsonian gauge couplings  $(g_{\alpha\beta}^{-2})^W$ ,  $(g_{\alpha 0}^{-2})^W$  and  $(g_{00}^{-2})^W$  are complicated non-holomorphic function of the moduli in accordance with eq. (3.8).

In supersymmetric EQFTs, holomorphic quantities are associated with chiral superspace integrals and consequently enjoy many no-renormalization theorems. In particular, in  $N = 2$  supersymmetric theories, the entire prepotential  $\mathcal{F}$  is not renormalized in any higher-loop order of the perturbation theory [1, 2]; in section 4.2 we present an independent argument for this no-renormalization theorem in the heterotic string context. Thus,

$$\mathcal{F} = \mathcal{F}^{(0)} + \mathcal{F}^{(1)} + \mathcal{F}^{(NP)}, \quad (3.39)$$

where  $\mathcal{F}^{(0)}$  is the tree-level prepotential,  $\mathcal{F}^{(1)}$  originates at the one-loop level of the EQFT while  $\mathcal{F}^{(NP)}$  is due to instantons and other non-perturbative effects (see refs. [3, 4] for a detailed analysis of such effects in rigid  $N = 2$  SSYM.); in this article we confine our attention to the purely perturbative properties of the prepotential  $\mathcal{F}$  and therefore drop the  $\mathcal{F}^{(NP)}$  term from our further discussion. In terms of the expansion (3.36), eq. (3.39) (without  $\mathcal{F}^{(NP)}$ ) means

$$\begin{aligned} h(\Phi) &= h^{(0)}(\Phi) + h^{(1)}(\Phi), \\ f_{ab}(\Phi) &= f_{ab}^{(0)}(\Phi) + f_{ab}^{(1)}(\Phi); \end{aligned} \quad (3.40)$$

in particular, for the non-Abelian gauge group factors  $G_{(a)}$ , the Wilsonian gauge couplings are

$$(g_{(a)}^{-2})^W = \text{Re } f_{(a)}^{(0)}(\Phi) + \text{Re } f_{(a)}^{(1)}(\Phi), \quad (3.41)$$

in complete analogy with the  $N = 1$  EQFTs.

Thus far we discussed the analytic properties of the Wilsonian couplings. Let us now turn to the physical, momentum-dependent effective gauge couplings  $g_{(a)}^{-2}(p^2)$  which account for all the quantum effects, both high-energy and low-energy. As argued in refs. [35, 36, 12, 13, 14], the low-energy effects due to light charged particles give rise to a non-holomorphic moduli dependence of these effective gauge couplings; a supersymmetric Ward identity  $\partial_\Phi g_{(a)}^{-2} = i\theta_{(a),\Phi}$  relates this non-holomorphicity to the non-integrability of the effective axionic couplings  $\theta_{(a),\Phi} \neq \partial_\Phi \theta_{(a)}(\Phi)$ . According to this Ward identity, the entire moduli dependence of the effective gauge couplings can be derived from that of the axionic couplings, which in turn follows from the connection terms proportional to  $\partial_\mu \Phi^\alpha$  in the Lagrangian for the charged fermions of the theory. For the moduli  $\Phi^\alpha$  belonging to  $N = 2$  vector multiplets, these connection terms have exactly the same form as in local  $N = 1$  supersymmetry, so that we may simply adapt the  $N = 1$  formula of [12] to our

present case. Thus, we find that to all orders in perturbation theory [11, 12, 13, 14],<sup>‡</sup>

$$\begin{aligned}
g_{(a)}^{-2}(\Phi, \bar{\Phi}, p^2) &= \text{Re } f_{(a)}(\Phi) + \frac{b_{(a)}}{16\pi^2} \log \frac{M_{\text{Pl}}^2}{p^2} + \frac{\sum_r n_r T_{(a)}(r)}{8\pi^2} K_{\Phi}(\Phi, \bar{\Phi}) \\
&+ \frac{T(G_{(a)})}{8\pi^2} \log g_{(a)}^{-2}(\Phi, \bar{\Phi}, p^2) - \frac{T(G_{(a)})}{8\pi^2} \log Z_{(a)}(\Phi, \bar{\Phi}, p^2) + \text{const},
\end{aligned} \tag{3.42}$$

where  $Z_{(a)}$  is the effective normalization factor for the scalar superpartners of the gauge bosons of  $G_{(a)}$  and we make use of an  $N = 2$  supersymmetric Ward identity that prevents similar normalization matrices for the hypermultiplets from depending on the moduli belonging to vector multiplets. Furthermore,  $N = 2$  supergravity provides for

$$Z_{(a)}(\Phi, \bar{\Phi}, p^2) = 4e^{K_{\Phi}(\Phi, \bar{\Phi})} g_{(a)}^{-2}(\Phi, \bar{\Phi}, p^2); \tag{3.43}$$

this relation is also valid to all orders of the perturbation theory as long as  $K_{\Phi}$  is derived from the quantum-corrected  $h(\Phi)$  rather than the tree-level  $h^{(0)}(\Phi)$ . Substituting (3.43) into (3.42), we arrive at

$$g_{(a)}^{-2}(\Phi, \bar{\Phi}, p^2) = \text{Re } f_{(a)}(\Phi) + \frac{b_{(a)}}{16\pi^2} \left( \log \frac{M_{\text{Pl}}^2}{p^2} + K_{\Phi}(\Phi, \bar{\Phi}) \right) + \text{const} \tag{3.44}$$

(for the Abelian effective gauge couplings  $g_{ab}^{-2}(\Phi, \bar{\Phi}, p^2)$ , read  $b_{ab} = 2 \sum_r n_r \text{Tr}_r(T_a T_b)$  for the  $b_{(a)}$ ). Although the above argument might suggest that the “constant” term in this formula could be a function of the moduli belonging to hypermultiplets, actually the effective gauge couplings are completely independent of any hypermultiplet moduli. For consistency’s sake, we have verified this statement by explicitly calculating the axionic couplings, but it can be better understood as a Ward identity of the  $N = 2$  supersymmetry, rigid or local: The  $N = 2$  gauge fields couple to *charged* hypermultiplets in a minimal gauge-covariant way (and hence the spectrum of such hypermultiplets affects the beta-function coefficients  $b_{(a)}$  in (3.44)), but they do not have two-derivative couplings to any *neutral* hypermultiplets and hence the gauge couplings cannot depend on the latter.

In  $N = 1$  supersymmetric gauge theories, the scalar normalization factors on the right hand side of eqs. (3.42) renormalize differently from those of the gauge fields; this leads to the higher-loop renormalization of the effective gauge couplings even though the Wilsonian gauge couplings renormalize only at the one-loop level [11] (or non-perturbatively). The extended  $N = 2$  supersymmetry eliminates this effect and hence the perturbative renormalization of both the Wilsonian and the effective gauge couplings stops at the one-loop level. Indeed, in the rigid case, the only difference between the two kinds of gauge couplings is the moduli-independent  $(b_{(a)}/16\pi^2) \log M_{\text{Pl}}^2/p^2$  term and the moduli dependence of both couplings can be described by the same holomorphic function  $f_{(a)}(\Phi)$ ; this behavior was important for the non-perturbative analysis of refs. [2, 3].

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<sup>‡</sup>In our notations,  $n_r$  is the number of charged hypermultiplets in the representation  $r$  of the gauge group,  $T_{(a)}(r)\delta^{ab} = \text{Tr}_r(T^a T^b)$  ( $T^a$  being the hermitian generators of the gauge group  $G_{(a)}$ ),  $T(G_{(a)})$  abbreviates  $T_{(a)}(\text{adjoint of } G_{(a)})$ , and  $b_{(a)} = 2 \sum_r n_r T_{(a)}(r) - 2T(G_{(a)})$  is the beta-function coefficient of the  $N = 2$  gauge theory. We presume  $p \ll M_{\text{Pl}}$ .

However, for the *locally*  $N = 2$  supersymmetric gauge theories, (3.44) tells us that although the effective gauge couplings do not renormalize at higher-loop orders of the perturbation theory, their moduli dependence is different from that of the corresponding Wilsonian couplings (unless  $b_{(a)} = 0$ ): In addition to a harmonic function  $\text{Re } f_{(a)}(\Phi)$ , the effective gauge coupling also contains a Kähler term. Note that this is exactly the behavior observed in the explicit string-loop calculation of the gauge couplings of the toroidal compactifications of  $(d = 6, N = 1)$  vacua in ref. [35], where the non-harmonic term in the string-threshold correction to the effective  $g_{(a)}^{-2}$  was found to be precisely  $b_{(a)}$  times the Kähler function of the toroidal moduli.

We conclude this section with a discussion of special points or subspaces of the moduli space where otherwise heavy particles become massless. For the sake of definiteness, let us assume that  $\phi$  is a spectrum-preserving modulus of the EQFT as long as  $|\phi| \ll 1$  but for  $\phi = 0$  the gauge group becomes enlarged because of additional massless vector multiplets, although the case of additional charged hypermultiplets can be handled in exactly the same way. Clearly, string vacua corresponding to  $|\langle\phi\rangle| \ll 1$  have to be described by a different EQFT and in that new EQFT the  $\phi$  scalar itself is no longer a spectrum-preserving modulus but a matter scalar  $C^a$  (or perhaps a linear combination of such  $C^a$ ). However, in the range of *moderately* small  $|\langle\phi\rangle|$ , both EQFTs are valid and should yield identical low-energy physical quantities. Therefore, we can use this overlap of the two EQFTs' domains of validity to relate their Wilsonian couplings to each other.

Physically, moderately small  $|\langle\phi\rangle|$  means that there is a threshold at the energy scale  $M_I \sim |\langle\phi\rangle|M_{\text{Pl}}$  that is well below the Planck scale but well above the scale one uses to measure the *low-energy* physical quantities. In this range, the difference between the small- $\phi$  EQFT and the large- $\phi$  EQFT is that the fields with  $O(M_I)$  masses are present in the former but are integrated out from the latter. Therefore, the difference between the Wilsonian gauge couplings of the two EQFTs is simply a threshold correction. Ref. [12] gives a formula for such threshold corrections for  $N = 1$  supersymmetric gauge theories and it is easy to see that it applies without any modifications in the present  $N = 2$  case.<sup>§</sup> Thus, to all orders of the perturbation theory,

$$f'_{(a)} - f_{(a)} = \frac{b'_{(a)} - b_{(a)}}{16\pi^2} \log \phi^2 + \text{const}, \quad (3.45)$$

where the primed quantities refer to the large- $\phi$  EQFT and the unprimed to the small- $\phi$  EQFT. Modulo a trivial change of notations, this relation also holds for the Abelian gauge couplings of the two EQFTs, including the gauge couplings of the vector member of the  $\phi$  supermultiplet itself. Since in the  $|\phi| \ll 1$  limit, the Wilsonian gauge coupling

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<sup>§</sup>Actually, the  $N = 2$  case is simpler because the masses of the short vector multiplets or hypermultiplets have to saturate the Bogomolnyĭ bound. (The long vector multiplets' net contribution to beta-functions is zero and hence they do not contribute to the threshold corrections either.) The threshold corrections to the Wilsonian gauge couplings involve the *unnormalized* masses of these multiplets [12], which are simply proportional to  $\phi M_{\text{Pl}}$  with coefficients that do not depend on any other moduli. This explains why the constant term in eq. (3.45) is indeed constant.

$(g_{\phi\phi}^{-2})^W$  of the large- $\phi$  EQFT is simply  $\frac{1}{2}\text{Re } \partial_\phi^2 h'$ , it follows that the  $\phi$  dependence of the moduli prepotential  $h'$  of that EQFT must have the form

$$h'(\phi, \Phi) = h(\Phi) - \frac{b_{(\phi)}}{16\pi^2} [\log \phi^2 + \text{const}] \phi^2 + f_{(\phi)}(\Phi) \phi^2 + O(\phi^3), \quad (3.46)$$

where  $b_{(\phi)}$  is the  $\beta$ -function coefficient of the gauge group under which  $\phi$  is charged at  $\phi = 0$ ,  $f_{(\phi)}$  is the corresponding Wilsonian coupling and the  $O(\phi^3)$  term is completely regular in the  $\phi \rightarrow 0$  limit (*i.e.*, it is a convergent power series  $A_3(\Phi)\phi^3 + A_4(\Phi)\phi^4 + \dots$ ). Later in this article (section 4.3), eq. (3.46) will help us to completely determine the one-loop moduli prepotential  $h^{(1)}(T, U)$  for toroidal compactifications of the six-dimensional heterotic string.

## 4 Low-energy $N = 2$ effective theories for Heterotic String Vacua

In the previous section we reviewed the couplings of  $N = 2$  vector multiplets at the classical and quantum level. In this section we study the effective Lagrangian of  $N = 2$  heterotic vacuum families and display the special properties of the prepotential  $F$  which arise in these theories.

### 4.1 Classical results

As we discussed in section 2, the dilaton and the antisymmetric tensor gauge field in  $N = 2$  heterotic compactifications are accompanied by an Abelian vector gauge field. Together they are contained in a new  $N = 2$  supermultiplet, called the vector-tensor multiplet, which is dual to an Abelian vector multiplet. The scalar component  $S$  of the latter includes the dilaton as its real part and the axion as its imaginary part. The couplings of the dilaton multiplet are independent of the properties of the internal SCFT and thus universal at the string tree level; in particular, the dilaton does not mix with any of the other scalar fields in the spectrum of the EQFT. Furthermore, the axion is subject to a continuous Peccei-Quinn symmetry, which implies that the Kähler potential is only a function of  $(S + \bar{S})$ . Both properties together imply that the moduli space contains the dilaton field  $S$  as the complex coordinate of a separate  $SU(1, 1)/U(1)$  factor. The only special Kähler manifold of any dimension  $n > 1$  that satisfies this constraint is the symmetric space [15]

$$\frac{SU(1, 1)}{U(1)} \otimes \frac{SO(2, n-1)}{SO(2) \times SO(n-1)}, \quad (4.1)$$

with a prepotential (up to symplectic reparametrizations)

$$F(X) = -\frac{X^1}{X^0} \left[ X^2 X^3 - \sum_{I=4}^n (X^I)^2 \right]. \quad (4.2)$$

The moduli  $\Phi$  of the previous section are identified with

$$S = -i\frac{X^1}{X^0}, \quad T = -i\frac{X^2}{X^0}, \quad U = -i\frac{X^3}{X^0}, \quad \phi^i = -i\frac{X^{i+3}}{X^0}, \quad (i = 1, \dots, P), \quad (4.3)$$

while the remaining  $X^I$  correspond to non-moduli scalars  $C^a = -iX^{a+P+3}/X^0$  ( $a = 1, \dots, (n = P - 3)$ ).  $T$  and  $U$  can be thought of as the toroidal moduli introduced in section 2. However, even in non-toroidal string vacua the prepotential  $F$  is given by (4.2), where  $X^2$  and  $X^3$  can be any two moduli.\* The moduli space (4.1) has been analyzed in detail in refs. [37, 10] The Kähler potential is easily calculated from eqs. (3.1), (4.2), (4.3) to be

$$K = -\log \left( (S + \bar{S}) \left[ (T + \bar{T})(U + \bar{U}) - \sum_i (\phi^i + \bar{\phi}^i)^2 - \sum_a (C^a + \bar{C}^a)^2 \right] \right). \quad (4.4)$$

In terms of our previous notation (eq. (3.36)–(3.41)) we have

$$\begin{aligned} h^{(0)} &= -S \left( TU - \sum_i (\phi^i)^2 \right), \quad f^{(0)} = S, \\ K_\Phi &= -\log(S + \bar{S}) - \log \left[ (T + \bar{T})(U + \bar{U}) - \sum_i (\phi^i + \bar{\phi}^i)^2 \right], \\ Z &= \frac{2}{(T + \bar{T})(U + \bar{U}) - \sum_i (\phi^i + \bar{\phi}^i)^2}. \end{aligned} \quad (4.5)$$

In particular, for any the non-Abelian factor in the gauge group  $G$  (or more generally any non-moduli vector multiplets) the tree-level gauge coupling is universal and depends only on the dilaton's vacuum expectation value,

$$g_{(a)}^{-2} = \text{Re } S \quad \text{for all } (a), \quad (4.6)$$

which is indeed a well-known tree-level property of the heterotic string.

On the other hand, the gauge couplings for the vector superpartners of the moduli scalars are given by the non-holomorphic matrix  $\mathcal{N}_{IJ}$ . Substituting eq. (4.2) into eq. (3.8),

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\*If the heterotic vacuum's spectrum contains only one modulus in addition to  $S$ , that modulus should be identified with  $(T + U)/2$  while the difference  $(T - U)/2$  is frozen at zero value. In terms of eq. (4.2), this means simply treating  $X^2 = X^3$  as a single independent coordinate. On the other hand, when the vacuum has no moduli at all besides  $S$ , the prepotential (4.2) is incompatible with non-zero gauge couplings for the non-moduli gauge fields and one *must* use coordinates for which  $F$  does not exist. However, we shall see momentarily that such coordinates are convenient for all heterotic  $N = 2$  vacua.

we find

$$\begin{aligned}
\mathcal{N}_{TT} &= -\frac{i}{2}e^{K_\Phi} (S + \bar{S})^2 (U + \bar{U})^2, & \mathcal{N}_{UU} &= -\frac{i}{2}e^{K_\Phi} (S + \bar{S})^2 (T + \bar{T})^2, \\
\mathcal{N}_{TU} &= i\bar{S} - \frac{i}{2}e^{K_\Phi} (S + \bar{S})^2 (T + \bar{T})(U + \bar{U}), \\
\mathcal{N}_{ij} &= -2i\bar{S}\delta_{ij} - 2ie^{K_\Phi} (S + \bar{S})^2 (\phi^i + \bar{\phi}^i)(\phi^j + \bar{\phi}^j), \\
\mathcal{N}_{Ti} &= ie^{K_\Phi} (S + \bar{S})^2 (U + \bar{U})(\phi^i + \bar{\phi}^i), & \mathcal{N}_{Ui} &= ie^{K_\Phi} (S + \bar{S})^2 (T + \bar{T})(\phi^i + \bar{\phi}^i), \\
\mathcal{N}_{SS} &= -\frac{i}{2}\frac{(U + \bar{U})(T + \bar{T}) - (\phi^j + \bar{\phi}^j)^2}{S + \bar{S}}, \\
\mathcal{N}_{ST} &= -\frac{i}{2}(U - \bar{U}), & \mathcal{N}_{SU} &= -\frac{i}{2}(T - \bar{T}), & \mathcal{N}_{Si} &= i(\phi^i - \bar{\phi}^i).
\end{aligned} \tag{4.7}$$

while the couplings of the graviphoton are

$$\begin{aligned}
\mathcal{N}_{00} &= -2i\bar{S}(\bar{T}\bar{U} - \bar{\phi}^{j2}) \\
&\quad + \frac{i}{2}e^{K_\Phi} [S(TU - \bar{T}\bar{U} - \phi^{j2} + \bar{\phi}^{j2}) - \bar{S}(\bar{T}U + T\bar{U} + 2\bar{T}\bar{U} - 2|\phi^j|^2 - 2\bar{\phi}^{j2})]^2, \\
\mathcal{N}_{0S} &= -\frac{1}{2}\frac{S(TU + \bar{T}\bar{U} - \phi^{j2} - \bar{\phi}^{j2}) - \bar{S}(T\bar{U} + \bar{T}U - 2|\phi^j|^2)}{S + \bar{S}}, \\
\mathcal{N}_{0T} &= -\bar{S}\bar{U} - \frac{1}{2}e^{K_\Phi} (S + \bar{S})(U + \bar{U}) \\
&\quad \times [S(TU - \bar{T}\bar{U} - \phi^{j2} + \bar{\phi}^{j2}) - \bar{S}(\bar{T}U + T\bar{U} + 2\bar{T}\bar{U} - 2|\phi^j|^2 - 2\bar{\phi}^{j2})], \\
\mathcal{N}_{0U} &= -\bar{S}\bar{T} - \frac{1}{2}e^{K_\Phi} (S + \bar{S})(T + \bar{T}) \\
&\quad \times [S(TU - \bar{T}\bar{U} - \phi^{j2} + \bar{\phi}^{j2}) - \bar{S}(\bar{T}U + T\bar{U} + 2\bar{T}\bar{U} - 2|\phi^j|^2 - 2\bar{\phi}^{j2})], \\
\mathcal{N}_{0i} &= 2\bar{S}\bar{\phi}^i + e^{K_\Phi} (S + \bar{S})(\phi^i + \bar{\phi}^i) \\
&\quad \times [S(TU - \bar{T}\bar{U} - \phi^{j2} + \bar{\phi}^{j2}) - \bar{S}(\bar{T}U + T\bar{U} + 2\bar{T}\bar{U} - 2|\phi^j|^2 - 2\bar{\phi}^{j2})].
\end{aligned} \tag{4.8}$$

We observe that all these gauge couplings are indeed non-holomorphic functions of the moduli, which is a direct consequence of the mixing between the graviphoton and the Abelian vector superpartners of the moduli scalars (*cf.* the second term in eq. (3.8)).<sup>†</sup> Furthermore, since  $e^{K_\Phi} \propto (S + \bar{S})^{-1}$ , one can easily see that most of the  $\mathcal{N}_{IJ}$  in eqs. (4.7) and (4.8) are proportional to the dilaton's expectation value and hence the corresponding gauge couplings become weak in the large-dilaton limit. The exceptions are  $\mathcal{N}_{SS}$ , which is proportional to  $S + \bar{S}$  and the off-diagonal matrix elements  $\mathcal{N}_{S(T,U,i \text{ or } 0)}$ , which are of the order  $O(1)$  in the large-dilaton limit. On the other hand, from the string theory we know

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<sup>†</sup>The non-holomorphicity of the tree-level gauge couplings is not present in  $N = 1$  supersymmetric orbifolds of the toroidal compactifications. Indeed, in such orbifolds the four  $U(1)$  gauge bosons related to  $T^2$  disappear from spectrum since the corresponding vertex operators are not invariant under the orbifold twist. Similarly, all possible Wilson-line moduli  $\phi^i$  of the compactification are not twist invariant and thus also disappear from the spectrum. Therefore, after the  $N = 1$  truncation, *all* the gauge couplings are given by simply  $S$ , which is the well-known property of the tree-level gauge coupling in  $N = 1$  heterotic vacua.



that *all* the *physical* low-energy couplings become weak in the large-dilaton limit, which suggests that the strongly-coupled  $F_{\mu\nu}^{+S}$  field strength in the dilaton  $N = 2$  superfield should be replaced with its dual (which is weakly coupled in the large-dilaton limit). In  $N = 2$  terms, this is achieved by the symplectic transformation  $(X^I, F_J) \rightarrow (\hat{X}^I, \hat{F}_J)$  where

$$\begin{aligned}\hat{X}^I &= X^I \quad \text{for } I \neq 1, & \hat{X}^1 &= F_1, \\ \hat{F}_I &= F_I \quad \text{for } I \neq 1, & \hat{F}_1 &= -X^1.\end{aligned}\tag{4.9}$$

The corresponding symplectic matrix  $\mathcal{O}'$ , defined by

$$\begin{pmatrix} \hat{X}^I \\ \hat{F}_J \end{pmatrix} = \mathcal{O}' \begin{pmatrix} X^K \\ F_L \end{pmatrix},\tag{4.10}$$

thus has nonzero elements

$$U'^I{}_J = V'^I{}_J = \delta^I_J \quad \text{for } I, J \neq 1, \quad Z'^{11} = 1, \quad W'_{11} = -1.\tag{4.11}$$

The new coordinates  $\hat{X}^I$  are, however, not independent, as they no longer depend on  $X^1$ . This reflects itself in the constraint

$$\eta_{IJ} \hat{X}^I \hat{X}^J \stackrel{\text{def}}{=} \hat{X}^1 \hat{X}^0 + \hat{X}^2 \hat{X}^3 - \hat{X}^i \hat{X}^i - \hat{X}^a \hat{X}^a = 0\tag{4.12}$$

(the first equality here defines the symmetric matrix  $\eta$ ), which can be easily verified by an explicit calculation.<sup>‡</sup> Consequently the matrix  $\mathcal{S}^I{}_J(X) = \partial \hat{X}^I / \partial X^J$  has zero determinant and hence no meaningful prepotential  $\hat{F}(\hat{X})$  can be defined [10]. Nevertheless, the gauge couplings and the Kähler potential for the moduli can be computed in the new basis from eq. (3.22). One finds [37, 10]

$$\begin{aligned}\hat{K}_\Phi &= K_\Phi = -\log(S + \bar{S}) - \log 2(\hat{z}^J \eta_{JI} \hat{z}^I) \\ \hat{\mathcal{N}}_{IJ} &= -2i\bar{S} \eta_{IJ} + 2i(S + \bar{S}) \frac{\eta_{IK} \eta_{JL} (\hat{z}^K \hat{z}^L + \hat{z}^K \hat{z}^L)}{\hat{z}^K \eta_{KL} \hat{z}^L},\end{aligned}\tag{4.13}$$

which has a rather symmetric form in terms of special coordinates  $\hat{z}^P \equiv \hat{X}^P / \hat{X}^0$ . In particular, in the new basis, all the  $\text{Im } \hat{\mathcal{N}}_{IJ}$  are proportional to  $S + \bar{S}$  and hence *all* the gauge couplings become weak in the large-dilaton limit. Note that the equality  $\hat{K}_\Phi = K_\Phi$  holds by virtue of the fact that the symplectic reparametrization (4.9) does not involve  $X^0$ .

The basis  $(\hat{X}^I, \hat{F}_J)$  is particularly well suited for the treatment of the target-space-duality symmetries of generic  $N = 2$  heterotic string vacua since the classical Lagrangian is manifestly invariant under symplectic transformations with  $\hat{W} = \hat{Z} = 0$  and  $\hat{U}$  (and thus  $\hat{V}$ ) belonging to  $SO(2, 2 + P)$ . This group follows from the requirement that the

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<sup>‡</sup>The  $\hat{F}_I$  are related to the  $\hat{X}^I$  according to  $\hat{F}_I = -2iS \eta_{IJ} \hat{X}^J$  and themselves satisfy a quadratic constraint  $\eta^{IJ} \hat{F}_I \hat{F}_J = 4\hat{F}_0 \hat{F}_1 + 4\hat{F}_2 \hat{F}_3 - \hat{F}_i \hat{F}_i - \hat{F}_a \hat{F}_a$ . However, unlike eq. (4.12), which remains valid in perturbation theory, the relations involving the  $\hat{F}_I$  are modified by the loop corrections. We shall return to this issue in section 4.2.

tensor  $\eta_{IJ}$  is left invariant:  $\hat{U}^T \eta \hat{U} = \eta$ . In that case the constraint (4.12) is manifestly invariant, while the Kähler metric and the gauge-coupling matrix  $\hat{\mathcal{N}}_{IJ}$  transform covariantly. Under this symmetry, the periods thus transform according to

$$\hat{X}^I \rightarrow \hat{U}^I{}_J \hat{X}^J, \quad \hat{F}_I \rightarrow [\hat{U}^{-1}]^J{}_I \hat{F}_J, \quad (4.14)$$

while the field strengths and vector potentials also transform according to the  $\hat{U}$  matrix. The dilaton field remains invariant at the classical level (see, however, the discussion in section 4.2).

Beyond the tree level, the continuous  $SO(2, 2+P)$  symmetry group of the low-energy effective theory is explicitly broken by the string loop corrections, but its maximal discrete subgroup  $SO(2, 2+P, \mathbf{Z})$  (or a subgroup thereof) remains an exact symmetry of the underlying string vacuum family [38] — the target-space duality of the Narain lattice — and hence should be manifest in the low-energy EQFT as well. In the following sections we shall discuss the constraints imposed by this discrete symmetry upon the loop corrections to the holomorphic prepotential  $\mathcal{F}$ . For the moment, let us simply make a few comments regarding the target-space duality group for toroidal compactifications. In that case this group contains  $PSL(2, \mathbf{Z})_T \times PSL(2, \mathbf{Z})_U$  and the action of the target-space duality group on the moduli can be determined from the heterotic string compactification as described in section 2. We recall that the first factor of the discrete group  $PSL(2, \mathbf{Z})_T \times PSL(2, \mathbf{Z})_U$  of toroidal compactifications, acts on the moduli as

$$T \rightarrow \frac{aT - ib}{icT + d}, \quad U \rightarrow U - \frac{ic}{icT + d} \phi^i \phi^i, \quad \phi^i \rightarrow \frac{\phi^i}{icT + d}, \quad (4.15)$$

while  $S$  remains invariant. The corresponding symplectic matrices (in the basis  $(\hat{X}^I, \hat{F}_I)$ ), are given by

$$\hat{U} = \begin{pmatrix} d & 0 & c & 0 & 0 \\ 0 & a & 0 & -b & 0 \\ b & 0 & a & 0 & 0 \\ 0 & -c & 0 & d & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1}_P \end{pmatrix}, \quad \hat{V} = (\hat{U}^T)^{-1} = \begin{pmatrix} a & 0 & -b & 0 & 0 \\ 0 & d & 0 & c & 0 \\ -c & 0 & d & 0 & 0 \\ 0 & b & 0 & a & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1}_P \end{pmatrix}, \quad (4.16)$$

while  $\hat{W} = \hat{Z} = 0$ . These matrices can be obtained in many ways, but by far the easiest procedure is to straightforwardly embed the  $SL(2, \mathbf{Z})_T$  into the  $SO(2, 2)$  subgroup of the  $SO(2, 2+P)$  group of classical  $\hat{U}$  matrices. The two facts to remember is that the  $\hat{X}^I$  should transform linearly into each other and that the field  $U$  should be inert in the absence of  $\phi^i$ . Alternatively one may employ the nilpotent subgroups constructed in subsection 3.1.

Although the discussion thus far was confined to the moduli, at the classical level there is no essential difference between the moduli and the charged scalars  $C^a$ , as one can see directly from the  $F$ -function in (4.2). Therefore, the previous symmetry consideration can be easily extended to include the  $X^a$  which are inert under  $SO(2, P+2)$ . Consequently, the  $C^a$  transform according to

$$C^a \rightarrow \frac{C^a}{icT + d} \quad (4.17)$$

under the  $PSL(2, \mathbf{Z})_T$  transformations.

Beside the target-space duality transformations, the classical field equations of the effective low-energy theory (but not its Lagrangian) are invariant under the so-called  $S$ -duality transformations [39, 38]. These transformations form an  $SL(2, \mathbf{Z})$  group, which act on the dilaton field  $S$  according to

$$S \rightarrow \frac{aS - ib}{icS + d}, \quad a, b, c, d \in \mathbf{Z}, \quad ad - bc = 1 \quad (4.18)$$

while the other scalar fields  $z^2, \dots, z^n$  remain invariant. The corresponding  $Sp(8+2P, \mathbf{Z})$  matrices in the basis  $(\hat{X}^I, \hat{F}_I)$  are

$$\hat{U}_J^I = d \delta_J^I, \quad \hat{V}_J^I = a \delta_J^I, \quad \hat{W}_{IJ} = -2b \eta_{IJ}, \quad \hat{Z}^{IJ} = -\frac{1}{2}c \eta^{IJ}. \quad (4.19)$$

This result follows straightforwardly from the tree-level relation  $\hat{F}_I = -2iS\eta_{IJ}\hat{X}^J$ , which demonstrates yet another advantage of the  $(\hat{X}^I, \hat{F}_I)$  basis. Of course, the same result can be obtained in the original  $(X^I, F_I)$  basis as well in a variety of ways, for example, one may use two subgroups of the  $SL(2, \mathbf{Z})_S$ , one with  $a = d = 1, c = 0$ , the other with  $a = d = 1, b = 0$ , which together generate the entire  $SL(2, \mathbf{Z})_S$ , for which the respective transformations are precisely those corresponding to (3.32) and (3.34). As the symplectic matrices (3.32) and (3.34) are in the  $(X^I, F_I)$  basis, we must use  $\mathcal{O} = \mathcal{O}'\hat{\mathcal{O}}\mathcal{O}'^{-1}$  in order to reproduce eq. (4.19). Needless to say, however, is that all of the above results, for both the  $S$ -duality and the target-space dualities, can be independently derived from string theory [30, 31, 32].

Among the  $S$ -dualities (4.18), of particular interest is the shift  $S \rightarrow S - i$ , which affects no physical couplings except the  $\theta$  angles which are shifted by  $2\pi$ ; this is the *discrete* Peccei-Quinn symmetry, which we assume to be exact at the quantum level.<sup>§</sup> On the other hand, the transformation  $S \rightarrow 1/S$  interchanges all the electric and magnetic  $U(1)$  field strengths and inverts all the gauge couplings (cf. eq.(3.24)). Correspondingly, the electric and magnetic charges are also interchanged, which means that this duality mixes elementary string states with the non-perturbative solitons and therefore is of inherently non-perturbative nature. It is presently unknown which of the  $S$ -duality transformations are true symmetries of the quantum theory, but it is clear that quantum corrections to the holomorphic prepotential necessarily modify the explicit form (4.18) of such transformations. We expect the corresponding symplectic matrices to constitute a subgroup of the matrices (4.19).

## 4.2 Perturbative corrections

This section is about the perturbative corrections to the prepotential for heterotic string vacua. As we argued in section 3.2, at the quantum level the distinction between moduli

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<sup>§</sup>We apologize for a rather cavalier normalization of the gauge couplings. For the dilaton field  $S$  normalized in accordance with eq. (4.18) for the  $S$ -dualities, the conventionally normalized tree-level gauge couplings should be  $g_{(a)}^{-2} = 4\pi k_{(a)} \text{Re } S$ , where  $k_{(a)}$  is the level of the Kač-Moody algebra giving rise to the gauge group  $G_{(a)}$ .

and non-moduli scalars becomes important due to their very different renormalization behavior. Therefore, it proves convenient to expand  $\mathcal{F}$  around small  $C^a$  as in eq. (3.36); at the tree level the moduli-dependent coefficients of this expansion are determined in eq. (4.5) and one is left with

$$\begin{aligned} h &= -S(TU - \sum_i \phi^i \phi^i) + h^{(1)}(T, U, \phi^i), \\ f_{(a)} &= S + f_{(a)}^{(1)}(T, U, \phi^i). \end{aligned} \tag{4.20}$$

These formulæ, in which both  $h^{(1)}$  and  $f^{(1)}$  are functions of all the moduli *except*  $S$ , uses the fact that the dilaton serves as the loop-counting parameter of the heterotic string. For the same reason, any possible two-loop or higher-loop corrections would have to be proportional to negative powers of the dilaton and because of the continuous Peccei-Quinn symmetry (which persists to all orders in the perturbation theory), such corrections would have to involve the negative powers of the  $(S + \bar{S})$  combination rather than just  $S$ . On the other hand,  $\bar{S}$  clearly cannot appear in the holomorphic prepotential  $\mathcal{F}(\Phi)$  and hence in string theory, all perturbative corrections to the prepotential stop at the one-loop level, in full analogy to the field-theoretical expansion (3.39), which also terminates at the one-loop order. In the  $N = 1$  context, the same argument forbids two- or higher-loop corrections to the Wilsonian gauge couplings [34]; for  $N = 2$  this non-renormalization theorem is more powerful since the prepotential  $\mathcal{F}$  determines both the Wilsonian gauge couplings and the Kähler potential. Furthermore, by similar arguments it follows that the couplings of the hypermultiplets are not corrected at all at any loop order and hence the tree-level hyper-moduli space is the exact hyper-moduli space to all orders in perturbation theory.

Neither  $h^{(1)}(T, U, \phi^i)$  nor  $f_{(a)}^{(1)}(T, U, \phi^i)$  can be arbitrary functions of the moduli, since they should respect any exact duality symmetry a string vacuum might have. In the previous section we saw that the tree-level geometry of the moduli space is invariant under the  $SO(2, 2 + P)$  isometry group, and from string theory we know that transformations belonging to a discrete  $SO(2, 2 + P, \mathbf{Z})$  subgroup of this isometry group are in fact exact symmetries of string vacua to all orders in perturbation theory. The goal of this and the following sections is to find the precise conditions such exact symmetries impose on the holomorphic functions  $h^{(1)}$  and  $f_{(a)}^{(1)}$ . For the present section, we assume a completely generic  $N = 2$  vacuum family of the heterotic string and keep our discussion as general as possible. In the following section we then specialize to toroidal compactification.

At the quantum level of the effective field theory, the Wilsonian Lagrangian does not necessarily share the quantum symmetries; only the physical, effective couplings have to be invariant functions of the moduli. Let us therefore begin with the moduli multiplets, for which the Wilsonian gauge couplings are equal to the effective couplings and hence are invariant or rather covariant under the exact modular symmetries of the string theory. Suppressing the non-moduli vector multiplets from our notations, we write the holomorphic prepotential for the remaining homogeneous variables  $X^I$  ( $I =$

$0, 1, \dots, P+3)$  as

$$F(X) = H^{(0)}(X) + H^{(1)}(X), \quad (4.21)$$

where  $H^{(0)}(X)$  is the tree-level prepotential (4.2) while  $H^{(1)}(X) = -i(X^0)^2 h^{(1)}$  represents the one-loop contribution. Both functions are homogeneous of second degree and according to (4.20)  $H^{(1)}$  does not depend on  $X^1$ . However, the most convenient variables for our purpose are again  $\hat{X}^I$  and  $\hat{F}_I$ . Since the one-loop prepotential  $H^{(1)}$  does not depend on  $X^1$ , it follows that  $\hat{X}^1 = F_1$  is not modified by loop corrections. Hence, in the quantum theory the  $\hat{X}^I$  satisfy exactly the same constraint (4.12) as in the classical case. On the other hand, the relation between the  $\hat{F}_I$  and the  $\hat{X}^I$  is sensitive to the one-loop prepotential  $H^{(1)}$  and we now have

$$\hat{F}_I = -2iS \eta_{IJ} \hat{X}^J + H_I^{(1)}, \quad (4.22)$$

where  $H_I^{(1)} \equiv \partial H^{(1)}(X)/\partial X^I$  (cf. eqs. (4.9) and (4.21)). Obviously  $H_1^{(1)} = 0$ , so that all the  $\hat{F}_{I \neq 1}$  are modified by the quantum corrections, but  $\hat{F}_1$  keeps its classical value  $\hat{F}_1 = -X^1 = -iS\hat{X}^0$ . For the same reason,  $H_I^{(1)}\hat{X}^I = H_I^{(1)}X^I$ ; at the same time,  $H_I^{(1)}X^I = 2H^{(1)}$  because of the homogeneity of the function  $H^{(1)}(X)$ . Combining these two facts with eq. (4.22) and with the constraint (4.12), we arrive at

$$H^{(1)}(X) = \frac{1}{2}\hat{F}_I \hat{X}^I, \quad (4.23)$$

which expresses the one-loop prepotential directly in terms of the symplectic variables  $(\hat{X}^I, \hat{F}_I)$ .

In perturbative string theory, the moduli fields  $T$ ,  $U$  and  $\phi^i$  have fixed relations to their vertex operators and hence the transformation rules for these fields are completely determined at the tree level and are not corrected by the string loops. In terms of the  $(\hat{X}^I, \hat{F}_I)$  variables of the EQFT, this means that the  $\hat{X}^I$  should transform exactly as in the classical theory (cf. 4.14), without any perturbative corrections. On the other hand, the corresponding transformation rules (4.14) for the  $\hat{F}_I$  become modified at the one-loop level since the Lagrangian is no longer invariant. Instead the transformation rules have to generate discrete shifts in various  $\theta$  angles due to monodromies around semi-classical singularities in the moduli space where massive string modes become massless. We have anticipated this situation in eqs. (3.23): Instead of the classical transformation rules (4.14), in the quantum theory,  $(\hat{X}^I, \hat{F}_I)$  transform according to

$$\hat{X}^I \rightarrow \hat{U}^I_J \hat{X}^J, \quad \hat{F}_I \rightarrow \hat{V}_I^J \hat{F}_J + \hat{W}_{IJ} \hat{X}^J, \quad (4.24)$$

where

$$\hat{V} = (\hat{U}^T)^{-1}, \quad \hat{W} = \hat{V}\Lambda, \quad \Lambda = \Lambda^T \quad (4.25)$$

and  $\hat{U}$  belongs to  $SO(2, 2+P, \mathbf{Z})$ . Classically,  $\Lambda = 0$ , but in the quantum theory,  $\Lambda$  is an arbitrary real symmetric matrix, which should be integer valued in some basis (but not necessarily in the basis of  $\hat{X}^I$  defined in eqs. (4.9) and (4.3)) so that the ambiguities in the  $\theta$  angles are discrete ( $\delta\theta = \hat{W}\hat{U}^{-1} = \hat{V}\Lambda\hat{V}^T$ ). In particular, for a closed monodromy around a singularity,  $\hat{X}^I \rightarrow \hat{X}^I$  i. e.  $\hat{U} = 1$ , but  $\Lambda \neq 0$  and  $\hat{F}_I \rightarrow \hat{F}_I + \Lambda_{IJ}\hat{X}^J$ .

We recall that the prepotential itself is in general not invariant under a symmetry of the equations of motion corresponding to the effective action, as one can easily verify for the tree-level results of the previous section, but the period transformation rules are correctly induced by the transformations of the coordinates. Therefore substituting the period transformations (4.24) into eq. (4.23), one immediately obtains the corresponding transformation rule for  $H^{(1)}$ ,

$$H^{(1)}(\tilde{X}) = H^{(1)}(X) + \frac{1}{2}\Lambda_{IJ}\hat{X}^I\hat{X}^J. \quad (4.26)$$

Note that the dilaton field does not appear anywhere in this formula. To put the symmetry relation (4.26) in its proper context, it is important to keep in mind that  $H^{(1)}$  should have a logarithmic singularity whenever an otherwise massive string mode becomes massless; therefore, as an analytic function of  $(X^0, X^2, \dots, X^{P+3})$ ,  $H^{(1)}(X)$  is generally multi-valued. According to eq. (4.26), the ambiguities of  $H^{(1)}$  amount to quadratic polynomials in the variables  $\hat{X}^I$  with some discrete real coefficients\*; indeed, under a closed monodromy one generally has  $H^{(1)} \rightarrow H^{(1)} + \frac{1}{2}\Lambda_{IJ}\hat{X}^I\hat{X}^J$  even though the fields  $(X^0, X^2, \dots, X^{P+3})$  remain unchanged. However, modulo these ambiguities,  $H^{(1)}$  should be invariant under all the exact symmetries of the perturbative string theory. This is the main result of this section.

Let us now turn our attention to the dilaton field  $S$ . In perturbative string theory, the dilaton vertex and its superpartners have fixed relations to the vector-tensor multiplet. However, the duality relation between this vector-tensor multiplet and the Abelian vector multiplet containing  $S = -iX^1/X^0$  is not fixed but suffers from perturbative corrections in both string theory and field theory. Therefore, while the vector-tensor multiplet is inert under all the perturbative symmetries of the string's vacuum, the  $S$  field is only invariant classically but has non-trivial transformation properties at the one-loop level of the quantum theory. Indeed, using the relation  $X^1 = -\hat{F}_1 = -iS\hat{X}^0$ , it is easy to show that the transformation rules (4.24) imply

$$S \rightarrow \tilde{S} = S + \frac{i\hat{V}_1^J (H_J^{(1)} + \Lambda_{JK}\hat{X}^K)}{\hat{U}^0_I \hat{X}^I}, \quad (4.27)$$

which in turn is sufficient to assure the correct transformation properties of all the  $\hat{F}_I$  and not just the  $\hat{F}_1$ .<sup>†</sup> However, if one does not insist upon the dilaton field being a

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\*In terms of  $h^{(1)}(T, U, \phi)$ , the ambiguities are quadratic polynomials in variables  $1, iT, iU, i\phi^i$  and  $(TU - \sum \phi^2)$  with discrete *imaginary* coefficients. Note that in terms of  $T, U$  and  $\phi^i$ , such polynomials are quartic.

<sup>†</sup>Using a convenient identity  $(\partial\hat{X}^1/\partial X^I) = \delta_I^1 - 2\eta_{IJ}(\hat{X}^J/\hat{X}^0)$ , one can show that eq. (4.26) implies the following transformation rule for the first derivatives of  $H^{(1)}$ :

$$\tilde{H}_I^{(1)} = \partial\tilde{H}^{(1)}/\partial\tilde{X}^I = \hat{V}_I^J H_J + \hat{W}_{IJ}\hat{X}^J + 2i(\tilde{S} - S)\eta_{IJ}\hat{U}_K^J \hat{X}^K, \quad (4.28)$$

where the difference  $(\tilde{S} - S)$  is precisely as in eq. (4.27) (note that this difference does not depend on the dilaton itself but only on the other moduli). It is easy to see that this transformation rule is precisely what is needed to assure complete consistency between eqs. (4.22) for the  $\hat{F}_I$  and the transformation rules (4.24).

*special* coordinate of the  $N = 2$  supersymmetry, it is possible to define a modular-invariant dilaton-like complex field by simply shifting  $S$  by a function of the other moduli. Specifically,<sup>‡</sup>

$$S^{\text{inv}} = S + \frac{1}{2(P+4)} \left[ i\eta^{IJ} H_{IJ}^{(1)} + L \right], \quad (4.29)$$

where  $L$  is a holomorphic function of the moduli whose duality transformation rules amount to imaginary constant shifts

$$L \rightarrow L - i\eta^{IJ} \Lambda_{IJ}. \quad (4.30)$$

In the following subsection we shall see that such a function is necessary to keep  $S^{\text{inv}}$  finite.

In  $N = 1$  supersymmetric vacua of the heterotic string,  $S$  belongs to a chiral supermultiplet dual to a linear multiplet. The linear multiplet has a fixed relation to the string vertices and is therefore inert under all the perturbative symmetries, while the  $S$  field has to be constructed order by order in perturbation theory. However, in the  $N = 1$  case one is free to redefine  $S$  by adding to it an arbitrary holomorphic function of the other moduli. This ambiguity is inherent in the chiral multiplet–linear multiplet duality relation and one may use it to define a modular invariant chiral superfield for the dilaton according to some analogue of eq. (4.29). By contrast, the  $N = 2$  supersymmetry does not allow for non-linear redefinition of the vector supermultiplets. Therefore, while the modular-variant scalar  $S$  is a legitimate member of an  $N = 2$  vector multiplet, the modular-invariant  $S^{\text{inv}}$  is not and hence cannot be simply used in place of the  $S$ .

Finally, consider the non-moduli gauge couplings  $f_{(a)}(\Phi)$ . Clearly, the physical gauge couplings  $g_{(a)}^{-2}(p^2)$  have to respect all the exact symmetries a string vacuum might possess. The Wilsonian gauge couplings, however, are not always invariant because they act as local counterterms compensating for potential anomalies of some of the symmetries [12, 13, 14]. For the local  $N = 2$  supersymmetry, such anomalies are associated with non-trivial transformations of the Kähler function  $K_\Phi$  in eq. (3.44). Indeed, under a generic  $SO(2, P+2, \mathbf{Z})$  symmetry (4.24), the Kähler function (3.1) transforms according to

$$\begin{aligned} K_\Phi(\Phi, \bar{\Phi}) &\rightarrow K_\Phi(\Phi, \bar{\Phi}) + \log |\tilde{X}^0/X^0|^2 \\ &= K_\Phi(\Phi, \bar{\Phi}) + \log |\hat{U}^0{}_J \hat{X}^J / \hat{X}^0|^2. \end{aligned} \quad (4.31)$$

Hence, in order to keep the physical coupling  $g_{(a)}^{-2}$  in eq. (3.44) invariant, the Wilsonian coupling  $f_{(a)}$  should transform as

$$f_{(a)}(\Phi) \rightarrow f_{(a)}(\Phi) - \frac{b_{(a)}}{8\pi^2} \log(\hat{U}^0{}_J \hat{X}^J / \hat{X}^0). \quad (4.32)$$

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<sup>‡</sup>As usual,  $H_{IJ}^{(1)}$  denote the second derivatives  $\partial_I \partial_J H^{(1)}$ . Note that since  $H_1^{(1)} = 0$ , the contraction  $\eta^{IJ} H_{IJ}^{(1)}$  involves only  $I, J = 2, \dots, P+3$ . The transformation rules for the  $H_{IJ}^{(1)}$  can be derived in the same manner as the rules for the first derivatives  $H_I^{(1)}$  in the previous footnote (although the algebra is somewhat more complicated).

This one-loop modification of the modular transformation properties of the  $f_{(a)}$  is entirely analogous to the situation encountered in  $N = 1$  supersymmetry. However, while in the  $N = 1$  case one may keep the dilaton  $S$  invariant and attribute the entire anomaly to the one-loop gauge couplings  $f_{(a)}^{(1)}(\Phi)$ , in the present  $N = 2$  case one has to live with the dilaton  $S$  transforming according to eq. (4.27). Consequently, the transformation rule for the  $f_{(a)}^{(1)}(\Phi)$  is given by the difference between eqs. (4.32) and (4.27).

### 4.3 Toroidal Compactifications

Thus far our analysis was generic and applicable to any  $N = 2$  vacuum of the heterotic string. Let us now apply this general formalism to the concrete case of toroidal compactifications of six-dimensional  $N = 1$  string vacua. We begin by turning off all the Wilson-line parameters  $\phi^i$ ; more precisely, we consider the domain of small  $\langle \phi^i \rangle$  in which  $\phi^i$  act as matter fields  $C^a$  rather than moduli, and our goal is to compute the quantum corrections  $h^{(1)}$  and  $f_{(a)}^{(1)}$  as functions of the toroidal moduli  $T$  and  $U$ . For the case at hand, the target-space duality group is  $SO(2, 2, \mathbf{Z})$  consisting of the  $T \leftrightarrow U$  exchange and of the  $PSL(2, \mathbf{Z})_T$  and  $PSL(2, \mathbf{Z})_U$  dualities whose action is described by eqs. (4.15) and (4.16). Substituting these dualities (for  $\phi^i = 0$ ) into the general transformation laws (4.26)–(4.32) of the previous section, we find

$$\begin{aligned} T &\rightarrow \frac{aT - ib}{icT + d}, & U &\rightarrow U, \\ h^{(1)}(T, U) &\rightarrow \frac{h^{(1)}(T, U) + \Xi(T, U)}{(icT + d)^2}, \\ f_{(a)}(S, T, U) &\rightarrow f_{(a)}(S, T, U) - \frac{b_{(a)}}{8\pi^2} \log(icT + d) \end{aligned} \tag{4.33}$$

and a similar set of transformations (with  $T$  and  $U$  interchanged) for the  $PSL(2, \mathbf{Z})_U$ . The appearance of  $\Xi = \frac{i}{2} \Lambda_{IJ} \hat{X}^I \hat{X}^J / (\hat{X}^0)^2$  in these formulæ complicates the symmetry properties of the one-loop moduli prepotential, which would otherwise be a modular function of weight  $-2$  with respect to both  $T$  and  $U$  dualities. However,  $\Xi$  is a quadratic polynomial in the variables  $(1, iT, iU, TU)$  and hence  $\partial_T^3 \Xi = \partial_U^3 \Xi = 0$ ; also, it is a mathematical fact that the third derivative of a modular function of weight  $-2$  is itself a modular function of weight  $+4$  even though the derivative is ordinary rather than covariant. From these two observations, we immediately learn that  $\partial_T^3 h^{(1)}(T, U)$  is a single-valued modular function of weight  $+4$  under the  $T$ -duality and of weight  $-2$  under the  $U$ -duality and there are no anomalies in its modular transformation properties; the same is of course true for the  $\partial_U^3 h^{(1)}$ , with the two modular weights interchanged.

The exact analytic form of a modular function can often be completely determined from the knowledge of its singularities and its asymptotic behavior when  $T \rightarrow \infty$  or  $U \rightarrow \infty$ . It was argued in ref. [14] that the gauge couplings of an  $N = 1$  orbifold cannot grow faster than a power of  $T$  or  $U$  in any decompactification limit and the same



argument applies here to the one-loop prepotential  $h^{(1)}$  and any of its derivatives. Let us therefore consider the singularity structure of the  $h^{(1)}(T, U)$ .

The gauge couplings of the  $[U(1)_L]^2$  containing the vector partners of  $T$  and  $U$  become singular whenever there are additional massless particles charged under this group. As discussed in section 2, this happens along the complex lines  $T \equiv U$ , where the  $U(1)_L^2$  group is enlarged to an  $SU(2) \times U(1)^*$ ; when such lines intersect each other, the group is further enlarged to an  $SU(2) \times SU(2)$  (at  $T \equiv U \equiv 1$ ) or an  $SU(3)$  (at  $T \equiv U \equiv \rho = e^{2\pi i/12}$ ). However, for a fixed generic value of  $U$ , the only singularities in the complex  $T$ -plane (or rather half-plane  $\text{Re } T > 0$ ) are at  $T \equiv U$  while the points  $T \equiv 1 \not\equiv U$  and  $T \equiv \rho \not\equiv U$  are perfectly regular; the same is of course true for the singularities in the  $U$ -plane (or rather half-plane) when  $T$  is held fixed at a generic value. Furthermore, the singular part of the prepotential along the  $T = U$  line is completely determined by eq. (3.46), in which we should identify  $\phi = \frac{1}{2}(T - U)$ ,  $\Phi = \frac{1}{2}(T + U)$  and  $b_{(\phi)} = -4$  (for an  $SU(2)$  without any non-singlet hypermultiplets). Hence, for generic  $T$  or  $U$  but small  $T - U$ ,

$$h(T \approx U) = \frac{1}{16\pi^2} (T - U)^2 \log(T - U)^2 + \text{regular}, \quad (4.34)$$

although the “regular” term here is only regular when  $T \approx U \not\equiv 1, \rho$ . Note that  $h$  is singular but finite when  $T \approx U$ ; its third derivatives  $\partial_T^3 h = \partial_T^3 h^{(1)}$  and  $\partial_U^3 h = \partial_U^3 h^{(1)}$  have simple poles at that point and similar poles whenever  $T \equiv U \pmod{SL(2, \mathbf{Z})}$ . This fact, plus all the other properties of the functions  $\partial_{T,U}^3 h^{(1)}(T, U)$  we have stated above, allow us to uniquely determine<sup>†</sup>

$$\begin{aligned} \partial_T^3 h^{(1)} &= \frac{+1}{2\pi} \frac{E_4(iT) E_4(iU) E_6(iU) \eta^{-24}(iU)}{j(iT) - j(iU)}, \\ \partial_U^3 h^{(1)} &= \frac{-1}{2\pi} \frac{E_4(iT) E_6(iT) \eta^{-24}(iT) E_4(iU)}{j(iT) - j(iU)}. \end{aligned} \quad (4.35)$$

This formula obviously determines the function  $h^{(1)}(T, U)$  itself up to a polynomial  $\Xi$  that is at most quadratic in  $T$  and in  $U$ , but we are unfortunately unable to write that function in terms of familiar modular functions. However, it is easy to see that eqs. (4.35) imply

$$\partial_T \partial_U h^{(1)} = \frac{-1}{4\pi^2} \log(j(iT) - j(iU)) + \text{finite}, \quad (4.36)$$

which has a curious property that the coefficient of the logarithmic divergence is  $2/8\pi^2$  when  $T \equiv U \not\equiv 1, \rho$  but becomes  $4/8\pi^2$  when  $T \equiv U \equiv 1$  and  $6/8\pi^2$  when  $T \equiv U \equiv \rho$ ,<sup>‡</sup> in precise agreement with the number of the massive string modes that become massless in each case (respectively, 2, 4 and 6 vector multiplets). Indeed, in ref. [21], this selfsame

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\*As in section 2, by  $T \equiv U$  we mean that  $T$  and  $U$  are equal modulo an  $SL(2, \mathbf{Z})$  transformation.

<sup>†</sup>In eq. (4.35),  $\eta$  is Dedekind’s eta-function,  $E_4$  and  $E_6$  are the normalized Eisenstein’s modular forms of respective weights +4 and +6 and  $j$  is the modular invariant function  $j = E_4^3/\eta^{24}$ . The arguments of these functions are  $iT$  and  $iU$  because mathematicians’ conventions differ from the string-theoretical conventions used in this article.

<sup>‡</sup>The derivative of the  $j(iT)$  function has a zero when  $T \equiv 1$  and a double zero when  $T \equiv \rho$ .

property of the heterotic string's vacua was used to determine the singularity structure of the gauge couplings such as (4.36). In this article, however, we arrived at eqs. (4.35) and (4.36) by considering only the  $T \equiv U \not\equiv 1, \rho$  vacua and the special properties of the  $T \equiv U \equiv 1, \rho$  vacua emerged courtesy of mathematical properties of the modular functions. Nevertheless, it is nice to have our result confirmed by an unrelated string-theoretical argument.

Now consider the dilaton. As we discussed in the previous section, the special  $N = 2$  coordinate  $S$  for the dilaton field of a quantum theory is not modular invariant, but there is a non-special coordinate  $S^{\text{inv}}$  that is modular invariant. The difference between the two coordinates

$$\sigma(T, U) \stackrel{\text{def}}{=} S^{\text{inv}} - S = -\frac{1}{2}\partial_T\partial_U h^{(1)}(T, U) + \frac{1}{8}L(T, U) \quad (4.37)$$

(*cf.* eq. (4.29)) must be finite throughout the  $(T, U)$  moduli space since otherwise one would not be able to use the value of  $S^{\text{inv}}$  as a universal string-loop counting parameter. For the same reason,  $\sigma(T, U)$  should not grow faster than  $T$  or  $U$  in the decompactification limits  $T \rightarrow \infty$  and  $U \rightarrow \infty$ . Combining these restrictions with eq. (4.36) for the  $\partial_T\partial_U h^{(1)}$  and with the requirement (4.30) that  $L(T, U)$  should be modular invariant up to a constant imaginary shift, we immediately arrive at

$$L(T, U) = -\frac{1}{\pi^2} \log(j(iT) - j(iU)) + \text{const}, \quad (4.38)$$

which indeed shifts by an imaginary constant when  $T$  (or  $U$ ) circles a singular line  $T \equiv U$ . Notice that although eqs. (4.36) and (4.38) provide for the *finiteness* of the difference (4.37), it is nevertheless singular and multivalued and its derivatives  $\partial_{T,U}\sigma(T, U)$  diverge logarithmically when  $T \equiv U$ . The multi-valuedness of the difference (4.37) is particularly disturbing since it implies that  $S$  is not only subject to non-trivial modular transformations but is not even single-valued for given values of  $T$  and  $U$ .<sup>§</sup> While one should expect such multi-valuedness of  $S$  in strongly coupled non-perturbative gauge theories, it is rather surprising to discover it already at the one-loop level.

Next consider the Wilsonian gauge couplings  $f_{(a)}$  for the gauge groups that the toroidal compactification inherits from the six-dimensional theory, *i. e.*, for all the gauge groups other than  $U(1)_{L+R}^4$ . The modular transformation rule for these couplings is given by the last eq. (4.33), which has exactly the same form as its analogues for the  $N = 1$  factorizable orbifolds considered in ref. [14]. Consequently, for exactly the same reasons as in the  $N = 1$  case, we now have

$$f_{(a)}(S, T, U) = S^{\text{inv}} - \frac{b_{(a)}}{8\pi^2} \left( \log \eta^2(iT) + \log \eta^2(iU) \right) + \text{const}. \quad (4.39)$$

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<sup>§</sup>Note that according to the arguments of the previous section, the  $S^{\text{inv}}$  coordinate is invariant under all semi-classical symmetries of the perturbative string theory, including the monodromies that leave  $T$  and  $U$  invariant; in other words,  $S^{\text{inv}}$  is both single-valued and modular-invariant. On the other hand, the  $N = 2$  superfield  $S$  is multi-valued, but any possible ambiguity in  $S$  has to be a linear combination of the four  $i\hat{X}^I/\hat{X}^0 = (i, T, U, iTU)$ ; these are the only ambiguities allowed for the  $N = 2$  superfields.

Note, however, that this formula involves the modular-invariant coordinate  $S^{\text{inv}}$  for the dilaton rather than the  $N = 2$  special coordinate  $S$  that appears in the tree-level term in eq. (4.20). Therefore, in terms of the  $N = 2$  supermultiplets, the one-loop corrections to the gauge couplings are

$$f_{(a)}^{(1)}(T, U) = -\frac{b_{(a)}}{8\pi^2} \left( \log \eta^2(iT) + \log \eta^2(iU) \right) + \sigma(T, U). \quad (4.40)$$

The first term on the right hand side here, plus the Kähler correction according to eq. (3.44), together constitute precisely the non-universal string-threshold correction to the gauge couplings obtained via an explicit string-loop calculation in ref. [35]. The second term on the right hand side of eq. (4.40) amounts to a universal threshold correction. Such universal corrections were disregarded in ref. [35], but they are also obtainable from string-loop calculations; we shall return to this point momentarily.

Before that, however, let us consider the loop-corrected Kähler potential  $K_\Phi(S, T, U)$ . Substituting the prepotential (4.20) into eq. (3.38), we immediately obtain

$$K_\Phi(S, T, U) = -\log \left[ S + \bar{S} + V_{GS}(T, U) \right] - \log(T + \bar{T}) - \log(U + \bar{U}), \quad (4.41)$$

where

$$V_{GS}(T, U) = \frac{2(h^{(1)} + \bar{h}^{(1)}) - (T + \bar{T})(\partial_T h^{(1)} + \partial_{\bar{T}} \bar{h}^{(1)}) - (U + \bar{U})(\partial_U h^{(1)} + \partial_{\bar{U}} \bar{h}^{(1)})}{(T + \bar{T})(U + \bar{U})} \quad (4.42)$$

is the Green-Schwarz term [13] describing the mixing of the dilaton with the moduli  $T$  and  $U$ . In  $N = 1$  vacua of the heterotic string such mixing arises at all loop levels of the string theory (except the tree level, of course), but in the  $N = 2$  case it is completely determined at the one-loop level. The importance of the Green-Schwarz term has to do with the fact that in the vector supermultiplet formalism for the dilaton, the true loop-counting parameter of the heterotic string is neither  $S + \bar{S}$ , nor even  $S^{\text{inv}} + \bar{S}^{\text{inv}}$ , but rather

$$S + \bar{S} + V_{GS}(T, U) = S^{\text{inv}} + \bar{S}^{\text{inv}} + V_{GS}^{\text{inv}}(T, U), \quad (4.43)$$

( $V_{GS}^{\text{inv}}$  is defined by this equation), which is directly related to the scalar component of the vector-tensor multiplet (or linear multiplet in the  $N = 1$  case). Therefore, a direct one-string-loop calculation of the threshold corrections  $\Delta_{(a)}$  to the gauge couplings should be interpreted according to [14]

$$\left[ g_{(a)}^{-2}(p^2) \right]^{\text{one-loop}} = \text{Re } S^{\text{inv}} + \frac{1}{2} V_{GS}^{\text{inv}}(T, U) + \Delta_{(a)}(T, U) + \frac{b_{(a)}}{16\pi^2} \log \frac{M_{\text{string}}^2}{p^2}. \quad (4.44)$$

Hence, a direct string calculation of the universal part of all the  $\Delta_{(a)}$  would immediately yield the modular-invariant Green-Schwarz term  $V_{GS}^{\text{inv}}$ , or rather

$$\begin{aligned} \Delta^{\text{univ}} &= -\frac{1}{2} V_{GS}^{\text{inv}}(T, U) \\ &= -\frac{1}{4} \left[ \frac{2}{T + \bar{T}} - (\partial_T + \partial_{\bar{T}}) \right] \left[ \frac{2}{U + \bar{U}} - (\partial_U + \partial_{\bar{U}}) \right] (h^{(1)} + \bar{h}^{(1)}) + \frac{1}{16} (L + \bar{L}). \end{aligned} \quad (4.45)$$

The calculation itself will be presented in a forthcoming article by some of the present authors; the techniques we have used are rather similar to those of ref. [35, 36]. For the purposes of the present article, let us simply state that the result is a complicated integral, which can be shown to satisfy the modular-invariant differential equation

$$\begin{aligned} \left[ (T + \bar{T})^2 \partial_T \partial_{\bar{T}} - 2 \right] \Delta^{\text{univ}}(T, U) &= \left[ (U + \bar{U})^2 \partial_U \partial_{\bar{U}} - 2 \right] \Delta^{\text{univ}}(T, U) \\ &= \frac{1}{8\pi^2} \log |j(iT) - j(iU)|^2. \end{aligned} \quad (4.46)$$

From this equation, one may directly show that the  $\Delta^{\text{univ}}$  has to have the form (4.45), where  $L(T, U)$  is precisely as in eq. (4.38), while  $h^{(1)}(T, U)$  is a holomorphic function that transforms according to eqs. (4.33) and has no singularities except at  $T \equiv U \pmod{SL(2, \mathbf{Z})}$  where  $\partial_T \partial_U h^{(1)}$  has a logarithmic divergence (4.36). This information is in turn sufficient to derive eqs. (4.35) without any further field-theoretical input. In this way, it is possible to obtain the Green-Schwarz term for the toroidal compactifications of six-dimensional vacua of the heterotic string (and, subsequently, of the  $N = 1$  orbifolds of such compactifications) without using *any* special properties of the  $N = 2$  supersymmetry but relying only on the string theory and on the  $N = 1$  arguments of ref. [14].

We conclude this article with a brief discussion of the Wilson-line moduli  $\phi^i$  which deform a toroidal compactification of a six-dimensional vacuum and break some of its gauge symmetries. For the sake of notational simplicity, we concentrate on a deformation involving a single Wilson-line modulus which we denote as simply  $\phi$ ; the deformations involving several such moduli can be analyzed in a similar manner. The deformation reduces the gauge group  $G$  of the un-deformed theory to a subgroup  $G' \subset S$ ; for small values of  $\phi$  this reduction can be described in field-theoretical terms as a Higgs mechanism in which  $\phi$  plays the role of the Higgs field. However, for  $|\langle \phi \rangle| \not\ll 1$ , one should simply integrate out the massive fields from the low-energy EQFT; in the resulting “deformed” EQFT,  $\phi$  becomes a spectrum-preserving modulus. As discussed in section 3.2, for moderately small values of  $\phi$  the prepotential of this “deformed” EQFT is governed by the eqs. (3.45) and (3.46), which for the case at hand give us

$$\begin{aligned} h'(S, T, U, \phi) &= S(\phi^2 - TU) + h^{(1)}(T, U) + \sigma(T, U) \phi^2 \\ &\quad - \frac{b_{(\phi)}}{8\pi^2} [\log \phi + \log \eta^2(iT) + \log \eta^2(iU) + \text{const}] \phi^2 + \dots, \\ f'_{(a)}(S, T, U, \phi) &= S + \sigma(T, U) + \frac{b'_{(a)}}{8\pi^2} \log \phi \\ &\quad - \frac{b_{(a)}}{8\pi^2} [\log \phi + \log \eta^2(iT) + \log \eta^2(iU) + \text{const}] + \dots, \end{aligned} \quad (4.47)$$

where the primes denote the parameters of the deformed theory,  $b_{(a)}$  and  $b_{(\phi)}$  are the appropriate beta-function coefficients of the un-deformed theory for  $\langle \phi \rangle = 0$  and the functions  $h^{(1)}(T, U)$  and  $\sigma(T, U)$  also belong to the un-deformed theory (*cf.* eqs. (4.35), (4.37) and (4.38) and also eqs. (4.40) for the gauge couplings of the un-deformed theory).

The ‘ $\dots$ ’ in eqs. (4.47) stand for the sub-leading terms carrying higher powers of the Wilson-line modulus  $\phi$ . Such terms are severely constrained by the discrete modular symmetries of the string theory. In particular, several modular symmetries are common to all Wilson-line deformations of toroidal compactifications, namely the  $T$ -duality, the  $U$ -duality and the “parities”  $T \leftrightarrow U$  and  $\phi \rightarrow -\phi$ . The transformation rules for the  $T$ -duality  $SL(2, \mathbf{Z})_T$  of the deformed theory are

$$\begin{aligned} T &\rightarrow \frac{aT - ib}{icT + d}, & U &\rightarrow U - \frac{ic\phi^2}{icT + d}, & \phi &\rightarrow \frac{\phi}{icT + d}, \\ h^{(1)'}(T, U, \phi) &\rightarrow \frac{h^{(1)'}(T, U, \phi) + \Xi(T, U, \phi)}{(icT + d)^2}, \end{aligned} \quad (4.48)$$

$$f'_{(a)}(S, T, U, \phi) \rightarrow f'_{(a)}(S, T, U, \phi) - \frac{b'_{(a)}}{8\pi^2} \log(icT + d) + \text{const.}$$

(cf. eqs. (4.15) and (4.26)–(4.32)); similar transformation rules with  $U$  and  $T$  interchanged describe the  $U$ -duality  $SL(2, \mathbf{Z})_U$  while the parities  $T \leftrightarrow U$  and  $\phi \rightarrow -\phi$  leave the pre-potential invariant. Our task therefore is to solve for holomorphic functions  $h^{(1)'}(T, U, \phi)$  and  $f'_{(a)}(S, T, U, \phi)$  which satisfy these transformation rules exactly and whose small- $\phi$  limits are precisely as in eqs. (4.47). For that purpose, let us first define the following functions:

$$\begin{aligned} \Omega_4(T, U, \phi) &= \sum_{n=0}^{\infty} \frac{\phi^{2n+4}}{n!(n+3)!} (\partial_T^n E_4(iT)) (\partial_U^n E_4(iU)), \\ \Omega_6(T, U, \phi) &= \sum_{n=0}^{\infty} \frac{\phi^{2n+6}}{n!(n+5)!} (\partial_T^n E_6(iT)) (\partial_U^n E_6(iU)), \\ \Omega_{12}(T, U, \phi) &= \sum_{n=0}^{\infty} \frac{\phi^{2n+12}}{n!(n+11)!} (\partial_T^n \eta^{24}(iT)) (\partial_U^n \eta^{24}(iU)). \end{aligned} \quad (4.49)$$

These three functions are  $SO(2, 2, \mathbf{Z})$  invariant, holomorphic, non-singular throughout the  $(T, U, \phi)$  moduli space ( $(\text{Re } T)(\text{Re } U) > (\text{Re } \phi)^2$ ) and do not grow faster than powers of the moduli in any decompactification limit. In other words, they are modular forms of the  $SO(2, 2, \mathbf{Z})$  and furthermore, all such modular forms are polynomials or power series in the  $\Omega_4$ ,  $\Omega_6$  and  $\Omega_{12}$ . We also need three additional holomorphic functions:

$$\begin{aligned} \Omega_A &= \sum_{n=1}^{\infty} \frac{\phi^{2n}}{(n!)^2} n (\partial_T \partial_U)^n \log(j(iT) - j(iU)), \\ \Omega_L &= \sum_{n=0}^{\infty} \frac{\phi^{2n}}{(n!)^2} \left( (\partial_T \partial_U)^n + n (\partial_T \partial_U)^{n-1} \left[ \partial_T \partial_U, \log(\eta^2(iT) \eta^2(iU)) \right] \right) \log(j(iT) - j(iU)), \\ \Omega_H &= \sum_{n=0}^{\infty} \frac{\phi^{2n}}{(n!)^2} \left( \frac{1}{2} (n^2 - 6n + 2) (\partial_T \partial_U)^n \right. \\ &\quad \left. + n(n-1)(n-2) (\partial_T \partial_U)^{n-3} \left[ (\partial_T \partial_U)^3, \log(\eta^2(iT) \eta^2(iU)) \right] \right) h^{(1)}(T, U), \end{aligned} \quad (4.50)$$

where in the last definition  $h^{(1)}(T, U)$  is precisely as defined by eqs. (4.35) for the undeformed toroidal compactifications. These functions are singular along the critical  $T \equiv$

$U$  lines; also,  $\Omega_A(T, U, \phi)$  is modular invariant,  $\Omega_L(T, U, \phi)$  is modular invariant up to a constant imaginary shift while  $\Omega_H(T, U, \phi)$  transforms precisely as the one-loop deformed prepotential  $h^{(1)'}(T, U, \phi)$  should transform according to (4.48).

With the above definitions (4.49) and (4.50) we can now write the general solution for moduli prepotential and the gauge couplings of the deformed theory that have both the right modular transformation properties (4.48) and the right  $\phi \rightarrow 0$  limits (4.47):

$$\begin{aligned}
h^{(1)'}(T, U, \phi) &= \Omega_H - \frac{\phi^2}{8\pi^2} \left[ \Omega_L - \frac{(1 - 2\Omega_A)^2}{\Omega_A(1 - \Omega_A)} \log \frac{1 - 2\Omega_A}{1 - \Omega_A} \right] - \frac{b_{(\phi)} \phi^2}{96\pi^2} \log \Omega_{12} \\
&\quad + \sum_{k, \ell, m \geq 0} C_{k\ell m}^h \Omega_4^k \Omega_6^\ell \Omega_{12}^m \phi^2, \\
f'_{(a)}(S, T, U, \phi) &= S^{\text{inv}} + \frac{b'_{(a)}}{8\pi^2} \log \phi - \frac{b_{(\phi)}}{96\pi^2} \log \Omega_{12} \\
&\quad + \sum_{k, \ell, m \geq 0} C_{k\ell m}^{(a)} \Omega_4^k \Omega_6^\ell \Omega_{12}^m,
\end{aligned} \tag{4.51}$$

where

$$\begin{aligned}
S^{\text{inv}}(S, T, U, \phi) &= S + \frac{1}{10} \left[ L'(T, U, \phi) - 4\partial_T \partial_U h^{(1)'}(T, U, \phi) + \partial_\phi^2 h^{(1)'}(T, U, \phi) \right] \\
&= S + \sigma(T, U) + O(\phi^2) \quad (\text{for small } \phi) \\
\text{for } L(T, U, \phi) &= \frac{-1}{\pi^2} \left[ \Omega_L(T, U, \phi) + \log \frac{1 - 2\Omega_A}{1 - \Omega_A} \right] + \frac{b_{(\phi)}}{48\pi^2} \log \Omega_{12}(T, U, \phi)
\end{aligned} \tag{4.52}$$

and  $C_{k\ell m}^h$  and  $C_{k\ell m}^{(a)}$  are arbitrary complex constants. Note however that in the small  $\phi$  limit,  $\Omega_4 = O(\phi^4)$ ,  $\Omega_6 = O(\phi^6)$  and  $\Omega_{12} = O(\phi^{12})$ . Hence, the solution (4.51) uniquely determines the moduli-dependent gauge couplings  $f'$  up to terms of the order  $\phi^4$  or higher and the moduli prepotential  $h$  up to the  $O(\phi^6)$  or higher-order terms. Also note, that the lowest terms in the expansions (4.49), (4.50) and (4.51) agree with previous results [21, 23] on threshold corrections with Wilson lines.

The coefficients  $C_{k\ell m}^h$  and  $C_{k\ell m}^{(a)}$  cannot be determined by the  $SO(2, 2, \mathbf{Z})$  modular symmetries that preserve the un-deformed subspace  $\phi = 0$  of the deformed  $(T, U, \phi)$  moduli space. Instead, one should demand the correct transformation properties of the prepotential under the entire symmetry group of the deformed moduli space, namely  $SO(2, 3, \mathbf{Z})$ . In particular, all the gauge couplings should be periodic functions of  $\phi$ . Note however that the period of the Wilson-line modulus  $\phi$  depends on the particular modulus. To be precise, the orientation of the  $(\hat{X}^0, \dots, \hat{X}^4)$  basis corresponding to the physical moduli  $T, U$  and  $\phi$  of the deformed theory relative to the crystallographic basis of the discrete  $SO(2, 3, \mathbf{Z})$  symmetry depends on a particular Wilson-line deformation under consideration. Consequently, at the  $O(\phi^4)$  level, the dependence of the gauge couplings on the Wilson-line modulus  $\phi$  depends on a particular modulus and in models with several Wilson-line moduli  $\phi^i$ , the  $O(\phi^4)$  terms in the gauge couplings generally have non-trivial index structure (*i. e.*,  $f_{ijkl}^{(a)}(T, U) \phi^i \phi^j \phi^k \phi^l$ ).

In conclusion, the classical target-space duality symmetries together with informations about the singularity structure of the gauge couplings allowed us to completely determine the holomorphic prepotential for toroidal compactifications to all orders in the perturbation theory. (Up to the ambiguity encoded in the  $\Xi(T, U)$  term, which amounts to a field-independent ambiguity of some  $\theta$  angles.) This result leads to the Green-Schwarz mixing of the dilaton and the moduli in the Kähler potential, which can be independently confirmed by a direct string-loop computation. Our analysis extends to moderately small Wilson-line deformations of toroidal compactifications, for which we obtained the model-independent leading terms in the expansion of the prepotential into powers of the Wilson-line moduli. We believe that our results are useful for the eventual non-perturbative analysis of the  $N = 2$  vacua of the heterotic string along the lines of refs. [3, 4].

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## Appendix A.

### Dilaton- $B_{\mu\nu}$ $N = 2$ Vector-Tensor supermultiplet

As mentioned in section 2, the  $N = 2$  heterotic string compactification gives rise to a new supermultiplet consisting of a scalar  $\phi$  (corresponding to the dilaton), a rank-two tensor gauge field  $B_{\mu\nu}$ , a vector gauge field  $V_\mu$  and a doublet of Majorana spinors  $\lambda_i$ . These fields describe an on-shell supermultiplet of two spin-0, one spin-1 and four spin-

1/2 states, which are also described by an  $N = 2$  vector multiplet. Therefore we expect that the vector-tensor multiplet can be converted into a vector multiplet by means of a duality transformation.

The off-shell structure of the vector-tensor multiplet differs from that of the vector multiplet in several respects. Off-shell counting reveals that the 8+8 field components can only be realized as an off-shell supermultiplet in the presence of a central charge. On shell this central charge vanishes. In the context of *local* supersymmetry the central charge must be gauged, and for that one needs at least one Abelian vector multiplet (whose corresponding gauge field could coincide with the graviphoton). Although the central charge acts in a rather subtle way on the components of the vector-tensor multiplet, we expect that its coupling to  $N = 2$  supergravity can be constructed along the same lines as that for scalar (hyper)multiplets. Here we confine ourselves to the linearized treatment of the multiplet.

The bosonic fields given above comprise only seven degrees of freedom. The missing degree of freedom is provided by a real scalar auxiliary field, which we denote by  $D$ . The linearized transformation rules of the vector-tensor multiplet are as follows,

$$\begin{aligned}
\delta\phi &= \bar{\epsilon}^i \lambda_i + \bar{\epsilon}_i \lambda^i, \\
\delta V_\mu &= i\varepsilon^{ij} \bar{\epsilon}_i \gamma_\mu \lambda_j - i\varepsilon_{ij} \bar{\epsilon}^i \gamma_\mu \lambda^j, \\
\delta B_{\mu\nu} &= 2\bar{\epsilon}^i \sigma_{\mu\nu} \lambda_i + 2\bar{\epsilon}_i \sigma_{\mu\nu} \lambda^i, \\
\delta\lambda_i &= (\not{\partial}\phi - i\not{H})\epsilon_i - \varepsilon_{ij} (i\sigma \cdot F^- + D)\epsilon^j, \\
\delta D &= \varepsilon^{ij} \bar{\epsilon}_i \not{\partial}\lambda_j + \varepsilon_{ij} \bar{\epsilon}^i \not{\partial}\lambda^j.
\end{aligned} \tag{A.1}$$

We use the chiral notation employed in [40, 24], where, for spinor quantities, upper and lower  $SU(2)$  indices  $i, j, \dots$  denote chiral components. For the spinors used above the precise correspondence is

$$\begin{aligned}
\gamma_5 \lambda_i &= \lambda_i, & \gamma_5 \epsilon^i &= \epsilon^i, \\
\gamma_5 \lambda^i &= -\lambda^i, & \gamma_5 \epsilon_i &= -\epsilon_i.
\end{aligned}$$

The  $SU(2)$  indices are raised and lowered by complex conjugation. The quantities  $H^\mu$  and  $F_{\mu\nu}^\pm$  are the field strength of the tensor field and the (anti)selfdual field strengths of the vector field, defined by

$$\begin{aligned}
H^\mu &= \frac{1}{2} i \varepsilon^{\mu\nu\rho\sigma} \partial_\nu B_{\rho\sigma}, \\
F_{\mu\nu} &= F_{\mu\nu}^+ + F_{\mu\nu}^- = \partial_\mu V_\nu - \partial_\nu V_\mu.
\end{aligned} \tag{A.2}$$

They satisfy the Bianchi identities

$$\partial_\mu H^\mu = 0, \quad \partial^\mu F_{\mu\nu}^+ = \partial^\mu F_{\mu\nu}^-. \tag{A.3}$$

For completeness we also record their supersymmetry transformations

$$\begin{aligned}
\delta F_{\mu\nu}^- &= i\varepsilon^{ij} \bar{\epsilon}_i \not{\partial} \sigma_{\mu\nu} \lambda_j + i\varepsilon_{ij} \bar{\epsilon}^i \sigma_{\mu\nu} \not{\partial} \lambda^j, \\
\delta H^\mu &= -2i\bar{\epsilon}^i \sigma^{\mu\nu} \partial_\nu \lambda_i + 2i\bar{\epsilon}_i \sigma^{\mu\nu} \partial_\nu \lambda^i.
\end{aligned} \tag{A.4}$$



The supersymmetry algebra closes on the above fields. The anticommutator of two supersymmetry transformations leads to a general coordinate transformation, central charge transformations and gauge transformations on the vector and tensor gauge fields. The central charge transformations take the form

$$\begin{aligned}
\delta_z \phi &= -\frac{1}{2}(z + \bar{z}) D, \\
\delta_z V_\mu &= \frac{1}{2}(z + \bar{z}) H_\mu, \\
\delta_z B_{\mu\nu} &= iz F_{\mu\nu}^- - i\bar{z} F_{\mu\nu}^+, \\
\delta_z \lambda_i &= -\frac{1}{2}(z + \bar{z}) \varepsilon_{ij} \not{\partial} \lambda^j, \\
\delta_z D &= \frac{1}{2}(z + \bar{z}) \partial^2 \phi.
\end{aligned} \tag{A.5}$$

In the supersymmetry commutator  $[\delta(\epsilon_1), \delta(\epsilon_2)]$  the central-charge parameter  $z$  equals  $z = 4\bar{\epsilon}_2^i \epsilon_1^j \varepsilon_{ij}$ ; the general-coordinate transformation is given by  $\xi^\mu = 2(\bar{\epsilon}_2^i \gamma^\mu \epsilon_{1i} + \bar{\epsilon}_{2i} \gamma^\mu \epsilon_1^i)$ .

From the product of two vector-tensor multiplets, one constructs an  $N = 2$  linear multiplet with central charge. In components this linear multiplet is given by

$$\begin{aligned}
L_{ij} &= \bar{\lambda}_i \lambda_j + \varepsilon_{ik} \varepsilon_{jl} \bar{\lambda}^k \lambda^l, \\
\varphi^i &= -(\not{\partial} \phi + iH) \lambda^i + \varepsilon^{ij} (-i\sigma \cdot F^- + D) \lambda_j, \\
G &= (\partial\phi)^2 - H^2 - 2iH \cdot \partial\phi + (F^-)^2 - D^2 + \bar{\lambda}^i \not{\partial} \lambda_i + \bar{\lambda}_i \not{\partial} \lambda^i, \\
E_\mu &= 2H^\nu (F^+ + F^-)_{\nu\mu} - 2i\partial^\nu \phi (F^+ - F^-)_{\nu\mu} + 2D \partial_\mu \phi \\
&\quad + \varepsilon^{ij} \lambda_i \partial_\mu \lambda_j + \varepsilon_{ij} \lambda^i \partial_\mu \lambda^j,
\end{aligned} \tag{A.6}$$

From (A.5) it is straightforward to obtain the central charge transformations of the linear multiplet components. For example,

$$\begin{aligned}
\delta_z L_{ij} &= (z + \bar{z}) \varepsilon_{k(i} (\bar{\lambda}_{j)} \not{\partial} \lambda^k - \bar{\lambda}^k \not{\partial} \lambda_{j)}), \\
\delta_z G &= (z + \bar{z}) \partial^\mu [-D(\partial_\mu \phi + iH_\mu) + 2H^\nu F_{\nu\mu}^- - 2i\phi \partial^\nu F_{\nu\mu}^- \\
&\quad - \frac{1}{2} \varepsilon^{ij} \bar{\lambda}_i \partial_\mu \lambda_j - \frac{1}{2} \varepsilon_{ij} \bar{\lambda}^i \partial_\mu \lambda^j].
\end{aligned} \tag{A.7}$$

The second equation shows that the appropriate constraint for the linear multiplet is satisfied [40],

$$(z + \bar{z}) \partial_\mu E^\mu = -\delta_z (G + \bar{G}). \tag{A.8}$$

The real part of  $G$  yields a linearized supersymmetric Lagrangian for the vector-tensor multiplet

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - \bar{\lambda}^i \not{\partial} \lambda_i + \frac{1}{2}H^2 - \frac{1}{4}F^2 + \frac{1}{2}D^2. \tag{A.9}$$

The other components of the linear multiplet play a role when considering the invariant action in the background of a vector multiplet that gauges the central charge.

As far as its physical degrees of freedom are concerned, the action (A.9) describes the same states as the action for a vector multiplet. In components this is rather obvious,

as one can, by means of a duality transformation, convert the antisymmetric tensor field  $B_{\mu\nu}$  into a (pseudo)scalar field. The latter can be combined with the field  $\phi$  into a complex scalar field. At present it is not clear how to perform the duality transformation in a way that is manifestly supersymmetric off shell. The fact that only one of the two multiplets has a central charge would certainly be a nontrivial aspect of such a duality transformation. It is worth mentioning that there exists also an  $N = 2$  tensor multiplet, consisting of three scalars, an antisymmetric tensor gauge field, a doublet of spinors and a complex auxiliary field, which can be converted to a scalar (hyper)multiplet by a duality transformation [41]. Again, one of the two multiplets involved in the duality transformation has an off-shell central charge. Also in this case it is not yet known how to perform the duality transformation such that supersymmetry is manifest off shell. As discussed in the previous section, the  $N = 2$  tensor multiplet arises in Calabi-Yau compactifications of type-II superstrings.

In principle, by studying the supergravity and Chern-Simons couplings of the new multiplet, one should be able to elucidate the restrictions imposed on the dilaton- $B_{\mu\nu}$  system as described in the dual formulation in terms of a vector multiplet. This is an interesting topic, which deserves further study. The strategy of this paper is to work in the dual formulation and use all possible information from string theory to specify the couplings of the corresponding vector multiplet. Therefore the thrust of our work is on vector multiplets.

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